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Determinants by K-Theory

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I Introduction

The main goal of this thesis will be to develop the theory of determinants for certain operator algebras. We will focus on K-theoretic methods, some of which can be found in [1] and [2]. The advantage of using K-theory, instead of analytic methods, will be a very abstract point of view and broad applicability.

Roughly speaking, the following two observations are crucial for the whole theory:

1. The relative, algebraic and topological K-groups, as defined in section II, are related by an exact sequence.
2. As shown hereafter, the usual determinant on $\mathrm{GL}_n(\mathbb{C})$ can be recovered by integration of paths and an application of the matrix trace. This method can then be extended to a more general setting.

To get a feeling for what we want to do, let us consider the determinant:

$$\det : \mathrm{GL}_n(\mathbb{C}) \to \mathbb{C}^*.$$

This function is a group homomorphism and has the very nice property that:

$$\det(\exp X) = e^{\operatorname{tr} X} \qquad (X \in M_n(\mathbb{C})),$$

where $\operatorname{tr} : M_n(\mathbb{C}) \to \mathbb{C}$ is the matrix trace. Hence, for $x \in \mathrm{GL}_n(\mathbb{C})$, it follows that

$$\det(x) = e^{\operatorname{tr}(\log x)}.$$

Note that a logarithm always exists. Since x has finite spectrum, one can find an appropriate branch of logarithm to apply holomorphic functional calculus. Denote by R_1 the set of smooth paths into $\mathrm{GL}_n(\mathbb{C})$ starting at the identity. Consider the function

$$F : R_1 \to M_n(\mathbb{C}) : \sigma \mapsto \int_0^1 \sigma'(t)\sigma(t)^{-1}\, dt.$$

Observe that

$$F(\sigma) = \log(\sigma(1)),$$

if σ takes for example values in a sufficiently small neighborhood of the identity. This allows to define the logarithm via the usual power series. For such a path one has

$$\det(\sigma(1)) = e^{\operatorname{tr}(F(\sigma))} = \exp\left(\operatorname{tr} \int_0^1 \sigma'(t)\sigma(t)^{-1}\, dt\right).$$

Since $\mathrm{GL}_n(\mathbb{C})$ is path connected[1], we can ask ourselves if the following definition makes sense:

$$\boxed{\widetilde{\det}(x) \overset{\text{def}}{=} e^{\operatorname{tr}(F(\sigma))} = \exp\left(\operatorname{tr}\int_0^1 \sigma'(t)\sigma(t)^{-1}\,dt\right)} \qquad (x \in \mathrm{GL}_n(\mathbb{C})),$$

where $\sigma \in R_1$ is any path with $\sigma(1) = x$.
First, one can check that this definition does not depend on the chosen path. Given two paths σ and τ connecting the identity and x, the path $\sigma\tau^{-1}$ is a loop at the identity. We know that $\pi_1(\mathrm{GL}_n(\mathbb{C})) = \mathbb{Z}$, with generator

$$\gamma(t) = \begin{pmatrix} e^{2\pi i t} & 0 \\ 0 & \mathbb{1}_{n-1} \end{pmatrix}.$$

Since the integral $F(\sigma)$ does not depend on the homotopy class (which we will not prove here), one has $F(\sigma\tau^{-1}) = F(\gamma^m) = 2\pi i m \mathbb{1}_n$ for some $m \in \mathbb{Z}$. But the exponential is well defined modulo $2\pi i\mathbb{Z}$, so that

$$e^{\operatorname{tr}(F(\sigma))} = e^{\operatorname{tr}(F(\tau))}.$$

It follows that the definition of $\widetilde{\det}$ indeed makes sense. Furthermore, one can show that

$$\det = \widetilde{\det}.$$

More generally, we will look at the well known Fredholm- and Fuglede-Kadison determinant. The Fredholm determinant will be a homomorphism

$$\det : \{T \in \mathrm{Inv}(B(H)) : T - \mathbb{1} \in \mathscr{L}^1(H)\} \to \mathbb{C}/(2\pi i\mathbb{Z}),$$

where H is a separable infinite dimensional Hilbert space and $\mathscr{L}^1(H)$ denotes the trace-class operators inside $B(H)$.

The Fuglede-Kadison determinant will be a homomorphism

$$\det_\tau : \mathrm{Inv}(N) \to \mathbb{C}/(i\mathbb{R}),$$

where N is a finite von Neumann algebra with trace τ and $\mathrm{Inv}(N)$ the invertible elements in N. This determinant was originally defined in [3].

As a new result, we will obtain a determinant for semi-finite von Neumann algebras. In that case we get a homomorphism

$$\det_\tau : \{x \in \mathrm{Inv}(N) : x - \mathbb{1} \in \mathscr{L}^1_\tau(N)\} \to \mathbb{C}/(i\mathbb{R}),$$

[1] It is a fact that any continuous path with values in $\mathrm{GL}_n(\mathbb{C})$ is homotopic to a smooth path, also having values in $\mathrm{GL}_n(\mathbb{C})$. A detailed proof can be found in the appendix.

where (N, τ) is a semi-finite von Neumann algebra and $\mathscr{L}^1_\tau(N)$ its trace ideal.

Finally, I want to thank my parents and Leonie for supporting me and Arnoud van Rooij for commenting on my thesis and especially for being such a great teacher. I am also very grateful for the many enlightening discussions with Peter Hochs and Jens Kaad. Especially, I want to thank Jens for introducing me to the world of non-commutative geometry and showing me its beauty.

II K-theory for Operator Algebras

II.1 Preliminaries

II.1.1. A good reference for topological K-theory for operator algebras is [4]. In [5] C^*-algebras are treated in a very detailed way. However, we will introduce notations and develop some basics of operator K-theory.

II.1.2 Definition. For a complex algebra A, we define the set of idempotents by

$$\mathrm{Idem}(A) := \{e \in A : e^2 = e\}.$$

If A is unital, we denote the set of invertible elements by

$$\mathrm{Inv}(A) \qquad \text{or} \qquad \mathrm{GL}_1(A).$$

If A has no unit, we define the group

$$\mathrm{GL}_1(A) := \{g \in \mathrm{GL}_1(A^\sim) : g - \mathbb{1} \in A\} \subset \mathrm{GL}_1(A^\sim).$$

By $A^\sim$ we mean the unitization of A, which is $A \oplus \mathbb{C}$. We will denote elements $(a, 0)$ just as a and $(0, 1)$ by $\mathbb{1}$. Then $A^\sim$ becomes an algebra by

$$(a + \lambda)(b + \mu) := (ab + \mu a + \lambda b) + \lambda\mu.$$

Furthermore, if A is a Banach algebra, we make $A^\sim$ into a Banach algebra by endowing it with the following complete norm:

$$\|a + \lambda\| := \|a\|_A + |\lambda|\,.$$

If A is a $*$-algebra we define

$$\mathrm{Proj}(A) := \{p \in A : p^2 = p = p^*\}.$$

II.1.3 Definition. Let A be a Banach algebra. On $\mathrm{Idem}(A)$ we define three equivalence relations. We say that $e, f \in \mathrm{Idem}(A)$ are

- **similar** if and only if there exists some $g \in \mathrm{GL}_1(A)$ with $geg^{-1} = f$.
- **homotopic** if and only if there exists a continuous path of idempotents connecting e and f.
- **algebraically equivalent** if and only if there exist elements $a, b \in A$ with $ab = e$ and $ba = f$.

We denote these relations by $\sim_s$, $\sim_h$ resp. $\sim_a$. It is easy to see that all three relations are equivalence relations. The only non-trivial part is to check that $\sim_a$ is transitive. See [4, Corollary 4.2.3] for a proof of this fact.

II.1.4 Lemma. *For A a unital Banach algebra, we have*

$$\sim_h \quad \subset \quad \sim_s \quad \subset \quad \sim_a .$$

Proof. The first inclusion holds true, since close idempotents are similar. See [4, Prop. 4.3.2.]. If $geg^{-1} = f$ then it is clear that $ab = e$ and $ba = f$ with $a = eg^{-1}$ and $b = g$. ■

II.2 Matrix algebras

II.2.1 Definition. Let A be a Banach algebra. Let $\mathrm{GL}_n(A) := \mathrm{GL}_1(M_n(A))$. We then define the algebraic direct limits

- $M_\infty(A) := \varinjlim M_n(A)$,
- $\mathrm{GL}(A) := \varinjlim \mathrm{GL}_n(A)$.

As inclusions we take

$$i_{n,n+1} : M_n(A) \hookrightarrow M_{n+1}(A) : x \mapsto \mathrm{diag}(x, 0) = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$i_{n,n+1} : \mathrm{GL}_n(A) \hookrightarrow \mathrm{GL}_{n+1}(A) : g \mapsto \mathrm{diag}(g, \mathbb{1}) = \begin{pmatrix} g & 0 \\ 0 & \mathbb{1} \end{pmatrix}.$$

Elements in $M_\infty(A)$ are infinite matrices which have only finitely many non-zero entries. Elements in $\mathrm{GL}(A)$ are infinite matrices of the form

$$\mathrm{diag}(g, \mathbb{1}_\infty),$$

where $g \in \mathrm{GL}_n(A)$ for some $n \in \mathbb{N}$ and $\mathbb{1}_\infty = \mathrm{diag}(\mathbb{1}, \mathbb{1}, \cdots)$. Finally, we let

$$\mathrm{Idem}_\infty(A) := \mathrm{Idem}(M_\infty(A)) = \varinjlim \mathrm{Idem}(M_n(A)).$$

II.2.2 Remark. Note that $M_\infty(A)$ is not unital, even if A is unital.

II.2.3 Definition. Let A be a Banach algebra. Then $M_\infty(A)$ becomes a normed algebra by letting

$$\|x\| := \sum_{i,j=1}^{\infty} \|x_{ij}\|_A .$$

More precisely, one endows each $M_n(A)$ with the norm

$$\|x\|_{M_n(A)} := \sum_{i,j=1}^{n} \|x_{ij}\|_A$$

and computes the norm of an element x in $M_\infty(A)$ by taking the norm of x in a large enough $M_n(A)$.
The set $\mathrm{GL}(A)$ is a topological group, when equipped with the metric

$$d(g,h) := \|g-h\|_{M_\infty(A)} .$$

By $\mathrm{GL}(A)_0$ we will denote the path component of the identity.

II.2.4 Remark. So far we equipped the $M_n(A)$ with *some* standard norm. To compute the topological K-theory of Banach algebras, we will see in lemma II.2.6 that we may take any norm on $M_n(A)$, which satisfies a few natural requirements. These requirements will ensure that the first topological K-theory of A is independent of the chosen norm.
However, one should keep in mind that different norms on the $M_n(A)$ might produce different normed algebras $M_\infty(A)$. The following example demonstrates this.

II.2.5 Example. Take a sequence $\{x^{(n)}\}_{n=1}^\infty$ in $M_\infty(\mathbb{C})$ where $x^{(n)} \in M_n(\mathbb{C})$ is the diagonal matrix

$$x^{(n)} = \mathrm{diag}\left(1, \frac{1}{2}, \cdots, \frac{1}{n}\right).$$

If we equip the $M_n(\mathbb{C})$ with the max-norm we see that the sequence is Cauchy, since for $n < m$ we have

$$\max_{i,j} \left\| x_{i,j}^{(n)} - x_{i,j}^{(m)} \right\| = \frac{1}{n+1}.$$

If we equip the $M_n(\mathbb{C})$ with the sum-norm, we see that the sequence is not Cauchy, since

$$\sum_{i,j} \left\| x_{i,j}^{(n)} - x_{i,j}^{(m)} \right\| = \frac{1}{n+1} + \cdots + \frac{1}{m} \qquad (\star)$$

Since the sum

$$\sum_{i=1}^{\infty} \frac{1}{i}$$

is not convergent, it follows that the sequence in $(\star)$ cannot be Cauchy. Therefore, as normed algebras, the direct limits are different.

II.2.6 Lemma. *Let $(A, \|\cdot\|_A)$ be a Banach algebra. Consider the set $\mathscr{N}$ consisting of sequences $\{\alpha_n : M_n(A) \to [0,\infty)\}_{n\in\mathbb{N}}$, where each α_n is a norm on $M_n(A)$ such that*

1. *For all $n \in \mathbb{N}$, $x \in M_n(A)$ and $m \geq n$ we have*

   $$\alpha_m(i_{n,m}(x)) = \alpha_n(x).$$

   We thus require the structure maps to be isometries.

2. *For all $n \in \mathbb{N}$ and $a \in A$ we have*

   $$\|a\|_A = \alpha_n(e_{ij}(a)),$$

   where $e_{ij}(a)$ is the matrix having value a at position (i,j) and 0 elsewhere.

3. *For all $x \in M_n(A)$ we have*

   $$\|x_{ij}\|_A \leq \alpha_n(x).$$

If $\alpha = \{\alpha_n\} \in \mathscr{N}$ we get a well-defined norm

$$\|\cdot\|_\alpha : M_\infty(A) \to [0,\infty)$$

and a metric

$$d_\alpha : \mathrm{GL}(A) \times \mathrm{GL}(A) \to [0,\infty)$$

by

$$d_\alpha(g,h) := \|g-h\|_\alpha .$$

Then, for $\alpha, \beta \in \mathscr{N}$ it follows that the topological groups $(\mathrm{GL}(A), \|\cdot\|_\alpha)$ and $(\mathrm{GL}(A), \|\cdot\|_\beta)$ have the same path component of the identity.

Proof. We make use of the following technical fact (see A.1.12), which tells us the following: Given a continuous path

$$\sigma : [0,1] \to (\mathrm{GL}(A), \|\cdot\|_\alpha)$$

from $\mathbb{1}$ to some $g \in \mathrm{GL}(A)$, one can find a path $\tilde{\sigma}$ connecting $\mathbb{1}$ and g such that $\tilde{\sigma}$ takes values in $\mathrm{GL}_n(A)$ for some fixed $n \in \mathbb{N}$. But the requirements on the sequence α ensures that $\|\cdot\|_{\beta_n} \leq n^2 \|\cdot\|_{\alpha_n}$. In particular

$$d_\beta(\sigma(t), \sigma(s)) \leq n^2 d_\alpha(\sigma(t), \sigma(s)) \qquad (t, s \in [0,1]).$$

This shows that

$$\tilde{\sigma} : [0,1] \to (\mathrm{GL}(A), \|\cdot\|_\beta)$$

is a continuous path connecting $\mathbb{1}$ and g.
By interchanging the roles of α and β one sees that

$$(\mathrm{GL}(A), d_\alpha)_0 = (\mathrm{GL}(A), d_\beta)_0.$$

■

II.2.7 Definition. We want to define the similarity relation $\sim_s$ on $\mathrm{Idem}_\infty(A)$, for A a Banach algebra. For $x, y \in \mathrm{Idem}_\infty(A)$, denote

$$x \sim_s y$$

if and only if there exists an element $g \in \mathrm{GL}(A)$ with

$$gxg^{-1} = y.$$

This means that we declare two idempotent matrices to be similar if and only if they are similar in some eventually larger matrix algebra.

II.2.8 Lemma. *If* $x, y \in \mathrm{GL}(A)$, *then* $\mathrm{diag}(xy, 1), \mathrm{diag}(x, y)$ *and* $\mathrm{diag}(y, x)$ *lie in the same path component of* $\mathrm{GL}(A)$. *See [4, Prop. 3.4.1].*

II.2.9 Lemma. *Any commutator* $[x, y]$ *in* $\mathrm{GL}(A)$ *can be written as* $[g, h]$, *where* $g, h \in \mathrm{GL}(A)_0$.

Proof. Use lemma II.2.8 and the identity

$$xyx^{-1}y^{-1} = \underbrace{\begin{pmatrix} x & 0 & 0 \\ 0 & x^{-1} & 0 \\ 0 & 0 & \mathbb{1}_n \end{pmatrix}}_{g} \underbrace{\begin{pmatrix} y & 0 & 0 \\ 0 & \mathbb{1}_n & 0 \\ 0 & 0 & y^{-1} \end{pmatrix}}_{h} \begin{pmatrix} x^{-1} & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & \mathbb{1}_n \end{pmatrix} \begin{pmatrix} y^{-1} & 0 & 0 \\ 0 & \mathbb{1}_n & 0 \\ 0 & 0 & y \end{pmatrix}$$

■

II.2.10 Lemma (Whitehead's lemma)**.** *The commutator subgroup of* $\mathrm{GL}(A)$, *denoted by* $[\mathrm{GL}(A), \mathrm{GL}(A)]$, *equals* $E(A)$, *the subgroup generated by elementary matrices. An elementary matrix* $E_{ij}(a)$ *has ones on the diagonal and only one non-zero off-diagonal entry* a *at position* (i, j), *where* $i \neq j$. *See [6, Proposition 2.1.4]*

II.2.11 Lemma. *Let A be a unital Banach algebra. Then all relations $\sim_s, \sim_h$ and $\sim_a$ coincide on* $\mathrm{Idem}_\infty(A)$.

Proof. By lemma II.1.4, we already know that $\sim_h \subset \sim_s \subset \sim_a$. Combining [4, Prop. 4.3.1] and [4, Prop. 4.4.1] it follows that $\sim_a \subset \sim_h$. ■

II.2.12. So far, we worked with Banach algebras. If A is a C^*-algebra all results also hold for idempotents replaced by projections and invertibles replaced by unitaries. See [4, Section 4.6].

II.2.13. The following lemma can be found in [4, Prop. 4.6.2.]. Since we will need the proof at a later stage (cf. lemma VII.3.2) , we will give it here.

II.2.14 Lemma. *Assume A is a C^*-algebra and $e \in \mathrm{Idem}(A)$. Then e is similar to a projection, where the similarity is possibly established in $A^\sim$.*

Proof. Let $e \in \mathrm{Idem}(A)$. Define $a = e^* - e$. Since a^*a is positive, we see that $g := \mathbb{1} + a^*a$ is invertible. Note that

$$a^*a = (e^* - e)(e - e^*) = e^*e - e^* - e + ee^*.$$

It follows that $ea^*a = ee^*e - e = a^*ae$. Therefore, we have

$$eg = ge = ee^*e.$$

So e and g commute and hence also e and g^{-1}. Since g is positive, we also get that e^*, g, g^{-1} commute. Now, we can define

$$p := ee^*g^{-1}.$$

Note that $pe = e$ and $ep = p$. To see this, we compute

$$pe = ee^*g^{-1}e = ee^*eg^{-1} = egg^{-1} = e.$$

Clearly, $ep = p$, since e is idempotent. It is also true that $p \in \mathrm{Proj}(A)$.

$$p^2 = ee^*eg^{-1}e^*g^{-1} = ee^*g^{-1} = p$$

and

$$p^* = (g^*)^{-1}ee^* = g^{-1}ee^* = ee^*g^{-1} = p.$$

Now, let $h := \mathbb{1} - p + e$. The inverse of h is $\mathbb{1} - e + p$. This follows from the fact that $(e - p)^2 = e - ep - pe + p = e - p - e + p = 0$. Then, h gives the desired similarity

$$heh^{-1} = (\mathbb{1} - p + e)e(\mathbb{1} - e + p) = p.$$

■

II.3 Algebraic K-theory

II.3.1. In this subsection we want to define the algebraic K-theory of a Banach algebra A, which is the same as the K-theory of A viewed as a ring. A good reference for K-theory of rings is [6].

II.3.2 Definition. Assume A is a **unital** Banach algebra. We define

$$V(A) := \mathrm{Idem}_\infty(A)/\sim_s .$$

By means of lemma II.2.11 one also has that

$$V(A) = \mathrm{Idem}_\infty(A)/\sim_a = \mathrm{Idem}_\infty(A)/\sim_h .$$

Elements of $V(A)$ will be written as classes $[e]$.

II.3.3 Lemma. *The triple* $(V(A), +, [0])$ *is an abelian monoid, with addition given by*

$$[e] + [f] = [\mathrm{diag}(e, f)].$$

See [6, Theorem 1.2.3].

II.3.4 Definition. Let A be a unital Banach algebra. Then define

$$K_0(A) := G(V(A)).$$

By G we mean the Grothendieck functor. See for example [5, Appendix G] for a detailed discussion of this functor.

II.3.5. Recall that elements of $K_0(A)$ are formal differences of classes

$$[e] - [f] \qquad ([e], [f] \in V(A)),$$

where e and f are idempotents in $M_\infty(A)$. One should have in mind the construction of $(\mathbb{Z}, +)$ from $(\mathbb{N}, +)$ or $(\mathbb{Q}, \cdot)$ from $(\mathbb{Z}, \cdot)$.

II.3.6 Lemma. *The assignment*

$$A \mapsto K_0(A)$$

defines a functor from the category of unital Banach algebras to abelian groups. The morphisms are in both cases homomorphisms. If $\phi : A \to B$ *is a homomorphism, one has*

$$\phi_*([e] - [f]) = [\phi(e)] - [\phi(f)],$$

where $\phi_* := K_0(\phi)$ *and* $\phi(e)$ *is* ϕ *applied to* e *entrywise. See [4, 5.5.1.].*

II.3.7 Definition. Assume A is a non-unital Banach algebra. By applying K_0 to $\pi : A^\sim \to \mathbb{C} : a + \lambda \mapsto \lambda$, one gets

$$\pi_* : K_0(A^\sim) \to K_0(\mathbb{C}).$$

We then define

$$K_0(A) := \ker(\pi_* : K_0(A^\sim) \to K_0(\mathbb{C})).$$

II.3.8 Lemma. *One has $K_0(\mathbb{C}) \cong \mathbb{Z}$ via the isomorphism*

$$[e] - [f] \mapsto \mathrm{rank}(e) - \mathrm{rank}(f).$$

See [4, Example 5.1.3.].

II.3.9 Definition. By the first algebraic K-theory of a unital Banach algebra A we understand the abelian group

$$K_1^{\mathrm{alg}}(A) := \mathrm{GL}(A)/[\mathrm{GL}(A), \mathrm{GL}(A)] = \mathrm{GL}(A)^{\mathrm{ab}},$$

where ab means abelianization. Again, in an obvious way, we get a functor K_1^{alg}. If A has no unit, we define

$$K_1^{\mathrm{alg}}(A) := \ker(\pi_* : K_1^{\mathrm{alg}}(A^\sim) \to K_1^{\mathrm{alg}}(\mathbb{C})).$$

II.3.10 Remark. As shown in [6, Prop. 2.2.2], it is true that the determinant induces an isomorphism

$$\det : K_1^{\mathrm{alg}}(\mathbb{F}) \to \mathbb{F}^*,$$

where $\mathbb{F}$ is any (commutative) field. In particular, $K_1^{\mathrm{alg}}(\mathbb{C}) = \mathbb{C}^*$.

II.3.11 Remark. If A is unital one also has

$$K_1^{\mathrm{alg}}(A) = \mathrm{GL}(A)/E(A).$$

This is a consequence of Whitehead's lemma (lemma II.2.10). In particular, $K_1^{\mathrm{alg}}(A)$ is trivial, if each element in $\mathrm{GL}(A)$ can be row-reduced to the identity matrix.

II.3.12. For this thesis, we only need the algebraic K-functors in degree 0 and 1, but we want to mention that they are defined for each natural number. In contrast to topological K-theory Bott periodicity does not hold for algebraic K-theory. This makes it much harder to compute algebraic K-groups.

II.4 Relative K-Theory

II.4.1 Definition. Let A be a unital Banach algebra. From [7, 6.15] it follows that a realization of the first relative K-theory of A can be defined as follows: Let

$$R(A)_1 := \bigcup_{n=1}^{\infty} \bigcup_{0<\epsilon<1} \left\{ \sigma \in C^\infty([0,1], \mathrm{GL}_n(A)) : \begin{array}{c} \sigma(0) = \mathbb{1} \\ \forall t \in [0,\epsilon] : \sigma(t) = \mathbb{1} \\ \forall t \in [0,\epsilon] : \sigma(1-t) = \sigma(1) \end{array} \right\}.$$

In words: $R(A)_1$ consists of smooth paths σ such that $\sigma(1) = \mathbb{1}$ and $\sigma(t) \in \mathrm{GL}_n(A)$ for all $t \in [0,1]$ and some fixed $n \in \mathbb{N}$. Furthermore, σ is constant in a neighborhood of 0 and 1.
Now, let $\sim$ being the relation of homotopy with fixed endpoints. The homotopy should be of a similar type as the paths above. To be more precise: We say that H is a homotopy with fixed endpoints making σ and τ homotopic, if the following holds:

$$\left[\begin{array}{l} H(0,s) = \mathbb{1},\ H(1,s) = \sigma(1) = \tau(1), \\ H(t,0) = \sigma,\ H(t,1) = \tau, \\ H(t,-), H(-,s) \in R(A)_1. \end{array} \right.$$

It is easy to check that $\sim$ is an equivalence relation. Now, we get a group

$$R(A)_1/\sim .$$

The product is given by pointwise multiplication. The first relative K-theory of A is then defined by

$$K_1^{\mathrm{rel}}(A) := (R(A)_1/\sim)^{\mathrm{ab}},$$

where ab denotes the abelianization.

II.4.2 Remark. At a first glance, the above definition seems unnecessarily artificial but later it will make our life easier. The main point is the following: whenever we work with paths into $\mathrm{GL}(A)$ it is perfectly fine to assume the path to be smooth and taking values in some $\mathrm{GL}_n(A)$. One also could have defined $R(A)_1$ to consist of all continuous paths starting at $\mathbb{1}$ and taking values in $\mathrm{GL}(A)$. However, after passing to homotopy classes, one can always find a representative as in the above definition. The resulting homotopy is exactly of the required type. The existence of such a homotopy is proved in a technical lemma, which can be found in A.1.12.

II.4.3 Definition. For A a unital Banach algebra, we define

$$\pi_1(\mathrm{GL}(A), \mathbb{1}) := \{\sigma \in R(A)_1 : \sigma(1) = \mathbb{1}\}/\sim .$$

The product is given by concatenation of loops.

II.4.4 Remark. As for the first relative K-theory of A one sees that the concatenation of two loops in $R(A)_1$ is again an element of $R(A)_1$. This is ensured by the fact that the loops are constant in a neighborhood of 0 and 1. By our technical lemma, we see again that our definition of $\pi_1(\mathrm{GL}(A), \mathbb{1})$ coincides with the usual definition of the first fundamental group of $\mathrm{GL}(A)$ with base point $\mathbb{1}$.

II.4.5. If A is not unital, one can still talk about $R(A)_1$ and $\pi_1(\mathrm{GL}(A), \mathbb{1})$. This is due to the fact that our technical lemma is still true for non-unital Banach algebras. See A.1.13.

II.5 Topological K-theory

II.5.1 Definition. Let A be a Banach algebra. Define

$$K_0^{\mathrm{top}}(A) := K_0(A).$$

In the following, all (algebraic and topological) zeroth K-groups will therefore be denoted by $K_0(A)$.

II.5.2 Definition. For A a unital Banach algebra, we define the first topological K-group of A as

$$K_1^{\mathrm{top}}(A) := \mathrm{GL}(A)/\,\mathrm{GL}(A)_0.$$

Recall, that if A has no unit we defined

$$\mathrm{GL}(A) = \{g \in \mathrm{GL}(A^\sim) : g - \mathbb{1} \in M_\infty(A)\}.$$

This gives a functor K_1^{top} from Banach algebras to abelian groups. To see that $K_1^{\mathrm{top}}(A)$ is abelian, one may use lemma II.2.8.

II.5.3 Lemma. *For all $n \in \mathbb{N}$, one has that $\mathrm{GL}_n(\mathbb{C})$ is path connected. In particular,*

$$K_1^{\mathrm{top}}(\mathbb{C}) = \{0\}.$$

(See lemma VI.1.3 or [4, Example 8.1.2].)

II.5.4 Remark. In contrast to the first algebraic K-theory, one does not need to take a kernel in order to get the right definition of the first topological K-theory, if A is non-unital. This comes from the fact that one wants K_1^{top} to be a split exact functor. Since

$$0 \to A \to A^\sim \to \mathbb{C} \to 0$$

is a split exact sequence, one gets (in addition with lemma II.5.3) that $K_1^{\mathrm{top}}(A) \cong K_1^{\mathrm{top}}(A^\sim)$.

II.5.5 Definition. Define the **cone** and **suspension** of a Banach algebra A by

$$C(A) := \{f \in C([0,1] \to A) : f(0) = 0\}$$

and

$$S(A) := \{f \in C(A) : f(1) = 0\}.$$

Under pointwise operations and sup-norm, these are Banach algebras.

II.5.6 Definition. For A a Banach algebra, we define

$$K_n^{\mathrm{top}}(A) := K_0(S^n A).$$

Since $K_0(SA) \cong K_1^{\mathrm{top}}(A)$ (see [4, Theorem 8.2.2]), the definition also makes sense for $n = 1$.

II.5.7. As one can read in [4, Section 9.2] we also have that

$$K_2^{\mathrm{top}}(A) = K_1^{\mathrm{top}}(SA) = \pi_1(\mathrm{GL}(A), \mathbb{1}),$$

where the group operation of the fundamental group is given by pointwise multiplication. The next lemma allows us to define

$$K_2^{\mathrm{top}}(A) := \pi_1(\mathrm{GL}(A), \mathbb{1}),$$

where π_1 denotes the fundamental group having concatenation of loops as group operation.

II.5.8 Lemma. *Multiplication of paths and concatenation is the same, modulo homotopy with fixed endpoints. To be more precise: Let G be a topological group and σ_0, σ_1 be paths into G starting at the identity. Then there is a homotopy with fixed endpoints between $\sigma_0 \cdot \sigma_1$ and $\sigma_0 * \sigma_1$, where the latter is defined by the following formula:*

$$(\sigma_0 * \sigma_1)(t) := \begin{cases} \sigma_0(2t) & \textit{if } t \in [0, 1/2] \\ \sigma_0(1) \cdot \sigma_1(2t-1) & \textit{if } t \in [1/2, 1]. \end{cases}$$

Proof. For $i = 0, 1$ define continuous functions $f_i : [0,1] \times [0,1] \to [0,1]$ by

$$\left[\begin{array}{l} f_0(s,t) := \min(t(1+s), 1) \\ f_1(s,t) := \max(t(1-s) - s, 0) \end{array} \right.$$

One can now check, that

$$H(s,t) := \sigma_0(f_0(s,t))\sigma_1(f_1(s,t))$$

is a homotopy with fixed endpoints between $\sigma_0\sigma_1$ and $\sigma_0 * \sigma_1$. ■

II.6 Bott periodicity

II.6.1. Without proof we will state the Bott periodicity theorem. In combination with our definition of K_2^{top} it will be very useful for computational reasons.

II.6.2 Theorem. *Let A be a Banach algebra. Then, there is an isomorphism, called the Bott map,*

$$\beta_A : K_0(A) \to K_2^{\mathrm{top}}(A)$$

induced by the map $V(A) \to K_2^{\mathrm{top}}(A) : [e] \mapsto \gamma_e$, where γ_e is the loop given by

$$\gamma_e : S^1 \to \mathrm{GL}(A) : z \mapsto ze + \mathbb{1} - e.$$

See [4, Theorem 9.2.1].

II.7 Relative pairs of Banach algebras

II.7.1 Definition. Let $(A, \|\cdot\|_A)$ be a **unital** Banach algebra and $J \subset A$ be a (not necessarily closed) ideal. We call (J, A) a relative pair if and only if the following holds:

1. J is a Banach algebra in its own right. That means that J is endowed with a norm $\|\cdot\|_J : J \to [0, \infty)$ such that $(J, \|\cdot\|_J)$ is a Banach algebra.
2. For all $a, b \in A$ and $j \in J$ we have
$$\|ajb\|_J \leq \|a\|_A \|j\|_J \|b\|_A .$$
3. For all $j \in J$ we have
$$\|j\|_A \leq \|j\|_J .$$

II.7.2 Definition. Assume (J, A) is a relative pair. We then define the unitization of J to be the subalgebra $J + \mathbb{C} \cdot \mathbb{1}_A$ and equip it with the norm $\|j + \lambda \mathbb{1}_A\| := \|j\|_J + |\lambda|$. As Banach algebra, this is isomorphic to the unitization as already defined. We want to emphasize that $J^{\sim}$ does not inherit the norm coming from A. Recall that

$$\mathrm{GL}(J) := \{g \in \mathrm{GL}(J^{\sim}) : g - \mathbb{1} \in M_\infty(J)\}.$$

Defining

$$d(g, h) := \|g - h\|_{M_\infty(J)} \qquad (g, h \in \mathrm{GL}(J)),$$

$\mathrm{GL}(J)$ becomes a topological group.

II.7.3 Example. Let H be a separable Hilbert space. The standard example of a relative pair is then

$$(\mathscr{L}^1(H), B(H))$$

where $\mathscr{L}^1(H)$ denotes the trace class operators in $B(H)$. We will see more of that, when discussing the Fredholm determinant (cf. V.1.4).

II.8 K-theory for relative pairs

II.8.1 Definition. We define the first algebraic K-theory of a relative pair (J, A) by the abelian group

$$K_1^{\mathrm{alg}}(J, A) := \mathrm{GL}(J)/[\mathrm{GL}(A), \mathrm{GL}(J)].$$

By $[\mathrm{GL}(A), \mathrm{GL}(J)]$ we mean the group generated by elements of the form

$$[g, h] = ghg^{-1}h^{-1} \qquad (g \in \mathrm{GL}(A), h \in \mathrm{GL}(J)).$$

It is not hard to verify that $[\mathrm{GL}(A), \mathrm{GL}(J)]$ is a normal subgroup of $\mathrm{GL}(J)$ which contains the commutator subgroup of $\mathrm{GL}(J)$. Therefore, the quotient is abelian.

II.8.2 Definition. Let

$$q : R(J)_1 \to R(J)_1/\sim$$

be the quotient map, where $\sim$ is the relation of being homotopic with fixed endpoints. We then define the first relative K-theory of a relative pair (J, A) to be the abelian group

$$K_1^{\mathrm{rel}}(J, A) := (R(J)_1/\sim)/(F(J, A)_1/\sim).$$

By $F(J, A)_1/\sim$ we mean the image under the quotient map of the group generated by elements of the form

$$[\sigma, \tau] = \sigma\tau\sigma^{-1}\tau^{-1} \qquad (\sigma \in R(J)_1, \tau \in R(A)_1).$$

Indeed, for $\sigma \in R(J)_1$ and $\tau \in R(A)_1$, one has $[\sigma, \tau] \in R(J)_1$. To see this, note that

$$\begin{aligned}\|\sigma(t)\tau(t) - \sigma(s)\tau(s)\|_J &\leq \|\sigma(t)\tau(t) - \sigma(t)\tau(s)\|_J + \|\sigma(t)\tau(s) - \sigma(s)\tau(s)\|_J \\ &\leq \|\sigma(t)\|_J \|\tau(t) - \tau(s)\|_A + \|\sigma(t) - \sigma(s)\|_J \|\tau(s)\|_J .\end{aligned}$$

Here, one uses property (iii) of definition II.7.1. We thus see that $\sigma\tau \in R(J)_1$ and by a similar argument also $[\sigma, \tau] \in R(J)_1$.

II.8.3 Definition. The first topological K-theory of the pair (J, A) is defined by

$$K_1^{\mathrm{top}}(J) = \mathrm{GL}(J)/\,\mathrm{GL}(J)_0.$$

The second topological K-theory of the pair (J, A) is defined by

$$K_2^{\mathrm{top}}(J) = \pi_1(\mathrm{GL}(J), \mathbb{1}).$$

II.8.4 Remark. In contrast to the first algebraic and relative K-theory of the pair (J, A) the first and second topological K-theory of the pair do not depend on the enveloping Banach algebra A.

III Comparing algebraic and topological K-theory

III.1.1. In this subsection we fix a relative pair (J, A). The goal is to prove that there exists a short exact sequence

$$K_2^{\mathrm{top}}(J) \xrightarrow{\partial} K_1^{\mathrm{rel}}(J, A) \xrightarrow{\theta} K_1^{\mathrm{alg}}(J, A) \xrightarrow{q} K_1^{\mathrm{top}}(J) \longrightarrow 0,$$

with morphisms

- $\partial : K_2^{\mathrm{top}}(J) \to K_1^{\mathrm{rel}}(J, A) : [\gamma] \mapsto [t \mapsto \gamma(e^{2\pi i t})]$,
- $\theta : K_1^{\mathrm{rel}}(J, A) \to K_1^{\mathrm{alg}}(J, A) : [\sigma] \mapsto [\sigma(1)^{-1}]$,
- $q : K_1^{\mathrm{alg}}(J, A) \to K_1^{\mathrm{top}}(J) : [g] \mapsto [g]$.

III.1.2 Remark. We will explain briefly, why these homomorphisms are well-defined.
By lemma II.5.8 it follows that the group operation of $\pi_1(\mathrm{GL}(J), \mathbb{1})$ coincides with the pointwise product, which shows that ∂ is well-defined.
If a path σ in $K_1^{\mathrm{rel}}(J, A)$ is trivial, its endpoint is a finite product of elements in $[\mathrm{GL}(J), \mathrm{GL}(A)]$. Its inverse is hence an element in $[\mathrm{GL}(A), \mathrm{GL}(J)]$. This means that $\sigma(1)^{-1}$ is trivial in $K_1^{\mathrm{alg}}(J, A)$.
From Whitehead's lemma (lemma II.2.10) it follows that q is well-defined.

III.1.3 Theorem. *The sequence*

$$K_2^{\mathrm{top}}(J) \xrightarrow{\partial} K_1^{\mathrm{rel}}(J, A) \xrightarrow{\theta} K_1^{\mathrm{alg}}(J, A) \xrightarrow{q} K_1^{\mathrm{top}}(J) \longrightarrow 0$$

is exact.

Proof. The only non-trivial thing to check is exactness at $K_1^{\text{rel}}(A)$. It is clear that $\theta \circ \partial = 0$.
Now, let $[\sigma] \in K_1^{\text{rel}}(J, A)$ such that $\theta([\sigma]) = [\sigma(1)^{-1}]$ is trivial in $K_1^{\text{alg}}(J, A)$. This means that $\sigma(1)^{-1}$ is a finite product of commutators. Without loss of generality, we may assume that $\sigma(1)^{-1}$ is itself a commutator. From lemma II.2.9 it follows that $\sigma(1)^{-1} = [g, h]$, where $g \in \mathrm{GL}(A)_0$ and $h \in \mathrm{GL}(J)_0$. Hence, there are paths $g_t \in R(A)_1$ and $h_t \in R(J)_1$ starting at $\mathbb{1}$ and ending at g resp. h. It follows that

$$\tau := [g_t, h_t]$$

is a path from $\mathbb{1}$ to $\sigma(1)^{-1}$. Then $\gamma := \tau^{-1} \cdot \sigma$ is a loop at $\mathbb{1}$ and gets mapped via ∂ to σ since τ^{-1} is trivial in $K_1^{\text{rel}}(J, A)$. ∎

III.1.4 Remark. We thus see that the relative K-theory indeed compares the algebraic and topological K-groups. This sequence will be one of our main tools to define several determinants on operator algebras, for which the first topological K-theory is trivial.

III.1.5 Corollary. *Let A be a unital Banach algebra. Then the following sequence is exact:*

$$K_2^{\text{top}}(A) \xrightarrow{\partial} K_1^{\text{rel}}(A) \xrightarrow{\theta} K_1^{\text{alg}}(A) \xrightarrow{q} K_1^{\text{top}}(A) \longrightarrow 0.$$

Proof. This follows by considering the relative pair (A, A). ∎

IV The relative Chern character

IV.1 Preliminaries

IV.1.1. In this section we construct the so-called first relative Chern character of an ideal J sitting relatively in A, which means for a relative pair (J, A). The Chern character will be a homomorphism

$$\mathrm{ch}^{\text{rel}} : K_1^{\text{rel}}(J, A) \to \mathrm{HC}_0(J, A)$$

where $\mathrm{HC}_0(J, A)$ is the zeroth continuous cyclic homology group of the pair (J, A).

IV.1.2 Definition. Let A, B be Banach spaces. By $A \otimes B$ we denote the algebraic tensor product of A and B, which is a complex vector space. Now define the so-called **projective tensor product** $A \otimes_\pi B$ of A and B by the completion of $A \otimes B$ under the norm

$$\|\xi\|_\pi := \inf \left\{ \sum_{i=1}^{n} \|a_i\|_A \, \|b_i\|_B : \xi = \sum_{i=1}^{n} a_i \otimes b_i \right\}.$$

IV.1.3 Proposition. $\|\cdot\|_\pi$ *does indeed define a norm on the algebraic tensor product of A and B. See [5, Prop. T.3.5].*

IV.1.4 Corollary. *Let A, B be Banach algebras. Then $A \otimes_\pi B$ is a Banach algebra.*

Proof. By the universal mapping property of algebraic tensor products, one can make $A \otimes B$ into an algebra. The multiplication is induced by

$$(a \otimes b)(c \otimes d) = (ac) \otimes (bd).$$

It is easily verified that $\|\cdot\|_\pi$ is submultiplicative on $A \otimes B$. Therefore, the multiplication can be extended to $A \otimes_\pi B$. This shows that $A \otimes_\pi B$ is a Banach algebra. ∎

IV.1.5 Definition. Let (J, A) be a relative pair. We can then consider the Banach algebra

$$J \otimes_\pi A.$$

We define a continuous multiplication operator on it by

$$m : J \otimes_\pi A \to J : j \otimes a \mapsto ja.$$

To see that m is continuous, we really need the interplay between the norms on J and A. Let $\xi = \sum_k j_k \otimes a_k \in J \otimes A$. Then

$$\|m(\xi)\|_J \leq \sum_k \|j_k a_k\|_J \leq \sum_k \|j_k\|_J \|a_k\|_A .$$

Hence $\|m(\xi)\|_J \leq \|\xi\|_\pi$, which shows that m is bounded on $J \otimes A$. Hence, it extends uniquely to a bounded operator on $J \otimes_\pi A$.

IV.1.6 Definition. By the zeroth relative continuous cyclic homology of the pair (J, A) we understand the quotient

$$\mathrm{HC}_0(J, A) := J / \mathrm{Im}(b),$$

where $b : J \otimes_\pi A \to J$ is the boundary map induced by

$$j \otimes a \mapsto ja - aj = [j, a].$$

Since the multiplication is continuous, we see that b is continuous, as well. We view $\mathrm{HC}_0(J, A)$ simply as a quotient of two vector spaces, without further topological structure.

IV.2 Construction of the relative Chern character

IV.2.1. If σ is a path in $R(J)_1$ we know that there exists some $n \in \mathbb{N}$ such that

$$\sigma : [0,1] \to \mathrm{GL}_n(J)$$

is a smooth map. Since $M_n(J)$ is a Banach algebra, we can talk about derivatives and integrals. The derivative σ', also denoted by

$$\frac{d\sigma}{dt},$$

is a map with values in $M_n(J)$, because the scalar part of $\sigma(t)$ is constant (i.e. the identity matrix of size n). Since J is an ideal inside $J^\sim$, it follows that we have a smooth path

$$\sigma'\sigma^{-1} : [0,1] \to M_n(J) : t \mapsto \sigma'(t)\sigma(t)^{-1}.$$

This path can then be integrated, by usual means of Riemann integration. This gives an element

$$\int_0^1 \frac{d\sigma}{dt}\sigma^{-1}\, dt \in M_n(J).$$

IV.2.2 Lemma. *Let $\sigma_0, \sigma_1 \in R(J)_1$ be homotopic. Then,*

$$\mathrm{TR}\left(\int_0^1 \frac{d\sigma_1}{dt}\sigma_1^{-1}\, dt - \int_0^1 \frac{d\sigma_0}{dt}\sigma_0^{-1}\, dt\right) \in \mathrm{Im}(b).$$

This proves, that there exists a well-defined map

$$\mathrm{ch}^{\mathrm{rel}} : R(J)_1/\sim\, \to \mathrm{HC}_0(J,A) : \sigma \mapsto \mathrm{TR}\left(\int_0^1 \frac{d\sigma}{dt}\sigma^{-1}\, dt\right).$$

By TR *we mean the matrix trace, which is given by the sum of the diagonal entries.*

Proof. First, we prove the result for paths $\sigma_i : [0,1] \to \mathrm{GL}_1(J)$. Let $H : [0,1]\times[0,1] \to \mathrm{GL}_1(J) : (t,s) \mapsto H(t,s)$ be a homotopy with fixed endpoints between σ_0 and σ_1. This means that $H(t,j) = \sigma_j(t)$ for $j = 0,1$. Define

$$L(H) := -\int_0^1\int_0^1 \frac{\partial H}{\partial t}H^{-1} \otimes \frac{\partial H}{\partial s}H^{-1}\, dt\, ds.$$

This defines an element of $J \otimes_\pi A$. Applying the operator b to it we see that

$$b(L(H)) = -\int_0^1\int_0^1 \left[\frac{\partial H}{\partial t}H^{-1}, \frac{\partial H}{\partial s}H^{-1}\right]\, dt\, ds.$$

So we need to compute

$$\frac{\partial H}{\partial t}H^{-1}\frac{\partial H}{\partial s}H^{-1}-\frac{\partial H}{\partial s}H^{-1}\frac{\partial H}{\partial t}H^{-1}. \tag{1}$$

Since

$$HH^{-1}=\mathbb{1},$$

differentiating with respect to t gives (Leibniz rule)

$$\frac{\partial H}{\partial t}H^{-1}+H\frac{\partial H^{-1}}{\partial t}=0.$$

Multiplying with H^{-1} from the left gives

$$H^{-1}\frac{\partial H}{\partial t}H^{-1}=-\frac{\partial H^{-1}}{\partial t}. \tag{2}$$

Note that (2) is also true when t is replaced by s. Applying (2) to (1) gives

$$\left[\frac{\partial H}{\partial t}H^{-1},\frac{\partial H}{\partial s}H^{-1}\right]=-\frac{\partial H}{\partial t}\frac{\partial H^{-1}}{\partial s}+\frac{\partial H}{\partial s}\frac{\partial H^{-1}}{\partial t}. \tag{3}$$

We also have that

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\partial H}{\partial s}H^{-1}\right)-\frac{\partial}{\partial s}\left(\frac{\partial H}{\partial t}H^{-1}\right)\\
&=\frac{\partial H}{\partial t\partial s}H^{-1}+\frac{\partial H}{\partial s}\frac{\partial H^{-1}}{\partial t}-\frac{\partial H}{\partial s\partial t}H^{-1}-\frac{\partial H}{\partial t}\frac{\partial H^{-1}}{\partial s}\\
&=\frac{\partial H}{\partial s}\frac{\partial H^{-1}}{\partial t}-\frac{\partial H}{\partial t}\frac{\partial H^{-1}}{\partial s}\\
&=(3).
\end{aligned}$$

Note that we made use of the fact that the order of differentiation is irrelevant. By the fundamental theorem of calculus, we conclude

$$\begin{aligned}
b(L(H))&=-\int_0^1\int_0^1\frac{\partial}{\partial t}\left(\frac{\partial H}{\partial s}H^{-1}\right)-\frac{\partial}{\partial s}\left(\frac{\partial H}{\partial t}H^{-1}\right)\,dt\,ds\\
&=\int_0^1\int_0^1\frac{\partial}{\partial s}\left(\frac{\partial H}{\partial t}H^{-1}\right)\,ds\,dt-\int_0^1\int_0^1\frac{\partial}{\partial t}\left(\frac{\partial H}{\partial s}H^{-1}\right)\,dt\,ds\\
&=\int_0^1\left(\frac{\partial H}{\partial t}(t,1)H(t,1)^{-1}-\frac{\partial H}{\partial s}(t,0)H(t,0)^{-1}\right)\,dt\\
&\qquad-\int_0^1\left(\frac{\partial H}{\partial s}(1,s)H(1,s)^{-1}-\frac{\partial H}{\partial s}(0,s)H(0,s)^{-1}\right)\,ds\\
&=\int_0^1\frac{d\sigma_1}{dt}\sigma_1^{-1}\,dt-\int_0^1\frac{d\sigma_0}{dt}\sigma_0^{-1}\,dt.
\end{aligned}$$

The next to last line of our computation vanishes, since the homotopy is constant in the endpoints (i.e. when $t = 0$ and $t = 1$). This completes the proof in case the paths take values in $\mathrm{GL}_1(J)$.
Now, we prove the general case. Assume that we are given two paths $\sigma_i : [0,1] \to \mathrm{GL}_n(J) = \mathrm{GL}_1(M_n(J))$. We want to apply the previous reasoning to the relative pair $(M_n(J), M_n(A))$ [2]. As before, we have a boundary map

$$b_n : M_n(J) \otimes_\pi M_n(A) \to M_n(J).$$

The previous proof then shows that

$$\int_0^1 \frac{d\sigma_1}{dt}\sigma_1^{-1}\,dt - \int_0^1 \frac{d\sigma_0}{dt}\sigma_0^{-1}\,dt = b_n(L(H)).$$

Applying the matrix trace hence gives

$$\mathrm{TR}\left(\int_0^1 \frac{d\sigma_1}{dt}\sigma_1^{-1}\,dt - \int_0^1 \frac{d\sigma_0}{dt}\sigma_0^{-1}\,dt\right) = (\mathrm{TR}\circ b_n)(L(H)).$$

The desired result will then follow from the fact (see [8, Cor. 1.2.3]) that the following diagram commutes:

$$\begin{array}{ccc} M_n(J)\otimes_\pi M_n(A) & \xrightarrow{b_n} & M_n(J) \\ {\scriptstyle \mathrm{TR}_n}\downarrow & & \downarrow{\scriptstyle \mathrm{TR}} \\ J\otimes_\pi A & \xrightarrow[b]{} & J \end{array}$$

By TR_n we mean the generalized trace. ■

IV.2.3 Lemma. *The map* $\mathrm{ch}^{\mathrm{rel}} : R(J)_1/\sim\ \to \mathrm{HC}_0(J,A)$ *descends to a homomorphism on the first relative relative K-theory of the pair* (J,A).

Proof. **(I)** First, we show that $\mathrm{ch}^{\mathrm{rel}}$ is a homomorphism. Take paths σ_0 and σ_1. Then, as proved in lemma II.5.8, we see that

$$\sigma_0\sigma_1 \sim \sigma_0 * \sigma_1,$$

where $*$ denotes the concatenation of paths. So

$$\mathrm{ch}^{\mathrm{rel}}([\sigma_0\cdot\sigma_1]) = \mathrm{ch}^{\mathrm{rel}}([\sigma_0 * \sigma_1]).$$

[2] One can verify that if (J,A) is a relative pair, then $(M_n(J), M_n(A))$ is a relative pair, too.

But for $\sigma_0 * \sigma_1$ it is easy to calculate the integral

$$\int_0^1 \frac{d(\sigma_0 * \sigma_1)}{dt}(\sigma_0 * \sigma_1)^{-1}\, dt.$$

Split the integral at $t = \frac{1}{2}$ and use simple substitution rules. This will prove that

$$\mathrm{ch}^{\mathrm{rel}}([\sigma_0 * \sigma_1]) = \mathrm{ch}^{\mathrm{rel}}([\sigma_0]) + \mathrm{ch}^{\mathrm{rel}}([\sigma_1]).$$

(II) Now, we prove that $\mathrm{ch}^{\mathrm{rel}}$ descends to a map on $K_1^{\mathrm{rel}}(J, A)$. First note the following: If $\sigma \in R(A)_1$ and $\tau \in R(J)_1$, we have $\sigma\tau\sigma^{-1} \in R(J)_1$. Consider the homotopy

$$H(s,t) := \sigma(f(s,t))\tau(t)\sigma(f(s,t))^{-1}$$

with

$$f(s,t) := ts + 1 - s = s(t-1) + 1.$$

Note that this homotopy indeed has fixed endpoints. We have

$$H(s,0) = \sigma(1-s) \cdot \mathbb{1} \cdot \sigma(1-s)^{-1} = \mathbb{1}$$

and

$$H(s,1) = \sigma(1)\tau(1)\sigma(1)^{-1}.$$

This proves that

$$\sigma\tau\sigma^{-1} \sim \sigma(1)\tau\sigma(1)^{-1}.$$

We now have

$$\mathrm{ch}^{\mathrm{rel}}([\sigma\tau\sigma^{-1}]) = \mathrm{ch}^{\mathrm{rel}}([\sigma(1)\tau\sigma(1)^{-1}]) \tag{$\star$}$$

Define

$$j := \int_0^1 \frac{d\tau}{dt}\tau^{-1}\, dt.$$

Then $\mathrm{ch}^{\mathrm{rel}}(\tau) = \mathrm{TR}(j)$. It follows that

$$\begin{aligned}(\mathrm{TR} \circ b_n)(\sigma(1)j \otimes \sigma(1)^{-1}) &= \mathrm{TR}(\sigma(1)j\sigma(1)^{-1} - j)\\ &= \mathrm{ch}^{\mathrm{rel}}(\sigma(1)\tau\sigma(1)^{-1}) - \mathrm{ch}^{\mathrm{rel}}(\tau)\end{aligned} \tag{$*$}$$

A little computation is required to prove that

$$\int_0^1 \frac{d\xi}{dt}\xi^{-1}\, dt = \sigma(1)j\sigma(1)^{-1},$$

with $\xi = \sigma(1)\tau\sigma(1)^{-1}$. In particular, one has to use that the cojugation operator is bounded. Since $\mathrm{TR} \circ b_n = b \circ \mathrm{TR}_n$ we see that $(*)$ is trivial in $\mathrm{HC}_0(J, A)$, which means that

$$\mathrm{ch}^{\mathrm{rel}}([\tau]) = \mathrm{ch}^{\mathrm{rel}}([\sigma(1)\tau\sigma(1)^{-1}])$$

Combining this with $(\star)$, one gets

$$\mathrm{ch}^{\mathrm{rel}}([\sigma\tau\sigma^{-1}\tau^{-1}]) = \mathrm{ch}^{\mathrm{rel}}([\tau]) - \mathrm{ch}^{\mathrm{rel}}([\tau]) = 0.$$

This proves that the map $\mathrm{ch}^{\mathrm{rel}}$ vanishes on the subgroup $F(J, A)_1/\sim$ of $R(J)_1/\sim$ and hence defines a homomorphism on $K_1^{\mathrm{rel}}(J, A)$.

■

IV.2.4 Definition. Let A be a Banach algebra. We define

$$\mathrm{HC}_0(A) := A/\operatorname{Im}(b),$$

where

$$b : A \otimes_\pi A \to A : a \otimes b \mapsto ab - ba = [a, b].$$

By $A \otimes_\pi A$ we denote as usual the projective tensor product of A with itself. We then get the relative Chern character

$$\mathrm{ch}^{\mathrm{rel}} : K_1^{\mathrm{rel}}(A) \to \mathrm{HC}_0(A)$$

induced by

$$\sigma \mapsto \mathrm{TR}\left(\int_0^1 \frac{d\sigma}{dt}\sigma^{-1}\, dt\right),$$

where TR is the sum over the diagonal. These definitions are obtained by considering the relative pair (A, A). In particular lemma IV.2.3 shows that the $\mathrm{ch}^{\mathrm{rel}}$ exists.

IV.2.5. We have now two versions of the relative Chern character. One for relative pairs and one for the non-relative case. The first well be used for the Fredholm determinant and semi-finite von Neumann algebras. The second for finite von Neumann algebras.

V The Fredholm determinant

V.1 Trace class operators

V.1.1. Before we start to look at the Fredholm determinant, we recall some basic facts about trace class operators. A detailed discussion can be found in [9, Chapter 2 & 3].

V.1.2. Let H be an infinite dimensional separable Hilbert space. Let K be a self-adjoint compact operator. It is well known that there exists an orthonormal basis of H, which consists of eigenvectors for K. Furthermore, if $\{\lambda_n\}_{n=1}^{\infty}$ denotes the eigenvalues taking multiplicities into account, one may assume that $|\lambda_n| \downarrow 0$.

V.1.3 Definition. Let K be a compact operator. Then $|K| = (K^*K)^{1/2}$ is a positive compact operator. Denote by $s_n(K)$ the eigenvalues of $|K|$ arranged such that $s_1(K) \geq s_2(K) \geq \cdots$. We then define

$$\mathscr{L}^1(H) := \left\{ K \in K(H) : \sum_{n=1}^{\infty} s_n(K) < \infty \right\}.$$

V.1.4. We now collect some standard facts about $\mathscr{L}^1(H)$. See for instance [9].

- $\mathscr{L}^1(H)$ is a $*$-ideal inside $B(H)$,
- The function

$$\|\cdot\|_1 : \mathscr{L}^1(H) \to [0, \infty) : K \mapsto \sum_{n=1}^{\infty} s_n(K)$$

  is a norm,
- the space $(\mathscr{L}^1(H), \|\cdot\|_1)$ is a Banach algebra,
- $\|K\| \leq \|K\|_1$, for $K \in \mathscr{L}^1(H)$,
- $\|RKS\|_1 \leq \|R\| \, \|K\|_1 \, \|S\|$ for $R, S \in B(H)$ and $K \in \mathscr{L}^1(H)$.

Recalling definition II.7.1, we see that

$$(\mathscr{L}^1(H), B(H))$$

is a relative pair.

V.1.5 Definition. Let H be a separable Hilbert space with orthonormal basis $\{e_i\}_{i=1}^{\infty}$. We then define the operator trace by

$$\mathrm{tr} : \mathscr{L}^1(H) \to \mathbb{C}$$

by

$$\mathrm{tr}(T) := \sum_{i=1}^{\infty} \langle Te_i, e_i \rangle.$$

This is a positive bounded linear functional, which also satisfies

$$\operatorname{tr}(TS) = \operatorname{tr}(ST) \qquad (T \in \mathscr{L}_1(H), S \in B(H)).$$

By positive we mean that $\operatorname{tr}(K) \geq 0$ if K is a positive trace class operator. The next paragraph clarifies why tr has finite values on $\mathscr{L}^1(H)$.

V.1.6. The trace does not depend on the orthonormal basis and by Lidskii's theorem, we know that

$$\operatorname{tr}(|A|) = \sum_{n=1}^{\infty} s_n(A).$$

In particular, we see that $\|T\|_1 = \operatorname{tr}(|T|)$.

V.1.7 Definition. Since tr is tracial it vanishes on commutators and hence descends to a map on $\mathrm{HC}_0(\mathscr{L}^1(H), B(H))$. We denote

$$\tilde{\tau} := \operatorname{tr} \circ \operatorname{ch}^{\mathrm{rel}}.$$

This gives a homomorphism

$$\tilde{\tau} : K_1^{\mathrm{rel}}(\mathscr{L}^1(H), B(H)) \to \mathbb{C}.$$

V.1.8 Remark. In the previous definition, one has in fact to be a bit more careful. Since $J \otimes_\pi A$ is the projective tensor product on which b acts, one has to verify that

$$\operatorname{tr}(b(\xi)) = 0,$$

where $\xi \in J \otimes_\pi A$. Writing $\xi = \lim_n \xi_n$ with $\xi_n \in J \otimes A$ and using continuity of b it is thus sufficient to know that tr is continuous with respect to $\|\cdot\|_1$. From [9, Theorem 3.1] it follows that

$$|\operatorname{tr} T| \leq \operatorname{tr}(|T|) = \|T\|_1 \qquad (T \in \mathscr{L}^1(H)).$$

This proves in particular that $\operatorname{tr} : \mathscr{L}^1(H) \to \mathbb{C}$ is $\|\cdot\|_1$ continuous. Therefore, tr really descends to a map on $\mathrm{HC}_0(\mathscr{L}^1(H), B(H))$.

V.1.9. The next remark deals with inductive limits of Banach algebras. See for example [4, Section 3.3] for a discussion of this subject.
Let $\{e_n\}_{n=1}^{\infty}$ be a basis for our separable Hilbert space H. Define

$$p_n := \sum_{i=1}^{n} e_i \otimes e_i,$$

by which we mean the orthogonal projection of H onto the span of the first n basis vectors. Then, any matrix $X \in M_n(\mathbb{C})$ acts on this span by usual matrix multiplication. Consider the embedding

$$\phi_n : M_n(\mathbb{C}) \to B(H) : X \mapsto Xp_n$$

and define

$$A_n := \phi_n(M_n(\mathbb{C})).$$

Furthermore, equip each A_n with the trace norm. Then, the A_n form an increasing sequence of Banach spaces inside $B(H)$. One can now look at the Banach algebra inductive limit of the A_n, denoted by

$$\varinjlim A_n.$$

In this case, this is the closure of the union

$$\bigcup_{n=1}^{\infty} A_n$$

inside $B(H)$ with respect to the trace norm. The following lemma clarifies the situation.

V.1.10 Lemma. *The inclusions $i_n : A_n \to \mathscr{L}^1(H)$ together with the inclusions $i_{n,m} : A_n \to A_m$ (for $n \leq m$) satisfy*

$$i_m \circ i_{n,m} = i_n.$$

By the universal property of $\varinjlim A_n$, there exists a homomorphism

$$\eta : \varinjlim A_n \to \mathscr{L}^1(H).$$

In fact η is an isomorphism of Banach algebras.

V.1.11 Lemma. *The first topological K-theory of the Banach algebra $\mathscr{L}^1(H)$ is trivial.*

Proof. The K_1^{top} functor is continuous (cf. [4, Remark after Prop. 8.1.3]). Since the Banach algebra $\mathscr{L}^1(H)$ is the direct limit of the algebras $M_n(\mathbb{C})$ in trace norm, one sees that $K_1^{\text{top}}(\mathscr{L}^1(H))$ is trivial, since

$$K_1^{\text{top}}(M_n(\mathbb{C})) = K_1^{\text{top}}(\mathbb{C}) = \{0\}.$$

■

V.2 The determinant

V.2.1 Lemma. *Recall that there is a homomorphism*

$$\partial : K_2^{\mathrm{top}}(J) \to K_1^{\mathrm{rel}}(J, A) : [\gamma] \mapsto [t \mapsto \gamma(e^{2\pi i t})],$$

where $(J, A) = (\mathscr{L}^1(H), B(H))$. *Then*

$$\tilde{\tau}(\operatorname{Im} \partial) \subset 2\pi i \mathbb{Z}.$$

Proof. As one can read in lemma VII.3.2, it is true that every idempotent in $\mathscr{L}^1(H)$ is similar (by an element of $\mathrm{GL}_1(\mathscr{L}^1(H))$, to a projection in $\mathscr{L}^1(H)$. This means that every class $[e] \in V(\mathscr{L}^1(H))$ is represented by a projection. Using Bott periodicity for topological K-theory (cf. theorem II.6.2), it follows that we have to compute the Chern character for loops of the form γ_p, where $\gamma_p(z) = zp + \mathbb{1} - p$ and p a projection in $\mathscr{L}^1(H)$. Since p has finite trace, it is immediate that its range has finite dimension. It follows that $\mathrm{tr}(p) = n$, where $n = \dim(pH)$. With this, we have

$$\begin{aligned} \mathrm{ch}^{\mathrm{rel}}(\partial([\gamma_p])) &= \int_0^1 \frac{d\gamma_p(e^{2\pi i t})}{dt} \gamma_p(e^{-2\pi i t}) dt \\ &= 2\pi i \int_0^1 e^{2\pi i t} p(e^{-2\pi i t} p + \mathbb{1} - p)\, dt \\ &= 2\pi i p. \end{aligned}$$

Now, note that

$$\mathrm{tr}(2\pi i p) = 2\pi i\, \mathrm{tr}(p) \in 2\pi i\ \mathbb{N}.$$

In general, one has that any class in $K_2^{\mathrm{top}}(\mathscr{L}^1(H))$ can be represented by an element of the form $[\gamma_p][\gamma_q]^{-1}$ for certain projections $p, q \in M_\infty(\mathscr{L}^1(H))$. By the above calculation and the fact that the matrix trace TR is positive, it follows that

$$\tilde{\tau}(\operatorname{Im} \partial) \subset 2\pi i\ \mathbb{Z}.$$

■

V.2.2. Now, we are able to define a determinant on the first relative algebraic K-theory of the trace class operators $\mathscr{L}^1(H)$ sitting inside $B(H)$. Recall that

$$\mathrm{GL}_1(\mathscr{L}^1(H)) = \{T \in \mathrm{Inv}(B(H)) : T - \mathbb{1} \in \mathscr{L}^1(H)\}.$$

As it will turn out, on $\mathrm{GL}_1(\mathscr{L}^1(H))$ the determinant will agree with the Fredholm determiant, which usually defined by purely analytic means (cf. [9, Lemma 3.3]). In contrast to that, we define the determinant by purely algebraic and K-theoretic methods. The analytic part is hidden in the existence of the relative Chern-character $\mathrm{ch}^{\mathrm{rel}} : K_1^{\mathrm{rel}}(J, A) \to \mathrm{HC}_0(J)$ where (J, A) is the relative pair $(\mathscr{L}^1(H), B(H))$.

V.2.3 Theorem. *There exists a homomorphism*

$$\det : K_1^{\mathrm{alg}}(J,A) \to \mathbb{C}^* : [x] \mapsto e^{\tilde{\tau}([\sigma])}$$

where $\sigma : [0,1] \to \mathrm{GL}(J)$ *is any path with*

$$[\sigma(1)] = [x].$$

Proof. First, recall the comparison sequence of the pair $(J,A) = (\mathscr{L}^1(H), B(H))$:

$$K_2^{\mathrm{top}}(J) \xrightarrow{\partial} K_1^{\mathrm{rel}}(J,A) \xrightarrow{\theta} K_1^{\mathrm{alg}}(J,A) \xrightarrow{q} K_1^{\mathrm{top}}(J) \longrightarrow 0.$$

From lemma V.1.11, we know that $K_1^{\mathrm{top}}(J) = \{0\}$. Therefore, the sequence reduces to

$$K_2^{\mathrm{top}}(J) \xrightarrow{\partial} K_1^{\mathrm{rel}}(J,A) \xrightarrow{\theta} K_1^{\mathrm{alg}}(J,A) \longrightarrow 0\ .$$

Let $[g] \in K_1^{\mathrm{alg}}(J,A)$ and assume that $[\sigma_i] \in K_1^{\mathrm{rel}}(J,A)$ $(i = 0,1)$ are lifts of $[g]$. Then $[\sigma_0(1)][\sigma_1(1)^{-1}] = [\sigma_0(1)\sigma_1(1)^{-1}] = [(\sigma_0\sigma_1^{-1})(1)]$ is trivial in the first algebraic K-theory of (J,A). So $[\sigma_0\sigma_1^{-1}]$ lies in the kernel of θ and thus in the image of ∂. By lemma V.1.11 we see that

$$\tilde{\tau}([\sigma_0\sigma_1^{-1}]) \in 2\pi i\mathbb{Z}.$$

It follows that

$$\tilde{\tau}([\sigma_0]) \equiv \tilde{\tau}([\sigma_1]) \mod 2\pi i\mathbb{Z}.$$

We thus have proved that there is a well-defined map

$$\det : K_1^{\mathrm{alg}}(J,A) \to \mathbb{C}/(2\pi i\mathbb{Z})$$

which maps $[x]$ to the class of $\tilde{\tau}([\sigma])$, where $[\sigma]$ is any lift of $[x]$. Note that by exactness of the sequence, a lift always exists. Since

$$e^{\bullet} : \mathbb{C}/(2\pi i\mathbb{Z}) \to \mathbb{C}^* : z + 2\pi i\mathbb{Z} \mapsto e^z$$

is a group isomorphism, the result follows. ∎

V.2.4. Notation. For $g \in \mathrm{GL}_1(\mathscr{L}^1(H))$ we denote by $\det(g)$ an application of the map

$$\mathrm{GL}_1(\mathscr{L}^1(H)) \overset{\pi}{\twoheadrightarrow} K_1^{\mathrm{alg}}(\mathscr{L}^1(H), B(H)) \xrightarrow{\det} \mathbb{C}^*$$

to g, where π is the quotient map.

V.2.5 Corollary. *The determinant* det *has the following properties:*

(i) $\det(gh) = \det(g)\det(h)$, *for* g, h *in* $\mathrm{GL}_1(J)$.

(ii) $\det(XgX^{-1}) = \det(g)$, *for all* $g \in \mathrm{GL}_1(J)$ *and* $X \in \mathrm{Inv}(B(H))$.

(iii) $\det(e^T) = e^{\mathrm{tr}(T)}$, *for all* $T \in \mathscr{L}^1(H)$.

This proves in particular that det *equals the usual Fredholm determinant (cf. [9, Chapter 3]).*

V.2.6 Remark. In (iii) of corollary V.2.5, one has of course to know that $e^T \in \mathrm{GL}_1(\mathscr{L}^1(H))$ for trace class operators T. But this is an easy fact: Since

$$e^T - \mathbb{1} = \sum_{k=1}^{\infty} \frac{T^k}{k!}$$

converges in trace norm, it follows that $e^T - \mathbb{1}$ sits inside $\mathscr{L}^1(H)$. One can also see this with corollary VII.2.11.

VI Finite von Neumann algebras

VI.1 Preliminaries

VI.1.1. We want to quickly recall some basic facts about von Neumann algebras. Assume N is $*$-subalgebra of $B(H)$ such that $\mathbb{1}_H \in N$. Then, we say that N is a **von Neumann algebra** if and only if one of the following three equivalent statements is satisfied:

- N is weakly closed ($\overline{N}^{\mathrm{WOT}} = N$),
- N is strongly closed ($\overline{N}^{\mathrm{SOT}} = N$),
- $N'' = N$, where N'' denotes the double commutant of N.

It is due to von Neumann, that the double commutant of N equals the weak closure of N. See [10, Theorem 4.1.5].
For us it will in particular be important that von Neumann algebras are closed under bounded Borel functional calculus and polar decomposition.

VI.1.2 Lemma. *Let N be a von Neumann algebra and assume $u \in N$ is unitary. Then*

$$u = e^{ia}$$

for some self-adjoint $a \in N$ such that $\sigma(a) \subset [-\pi, \pi]$.

Proof. If $f : [-\pi, \pi) \to S^1 : t \mapsto e^{it}$, then f is a continuous bijection. However, the inverse is continuous at every point $z \in S^1 \setminus \{-1\}$. So $g = f^{-1}$ is a bounded Borel function. Define $a := g(u)$ by means of the bounded Borel functional calculus at u. We claim that $e^{ia} = u$. As proved in [10, Theorem 2.5.7] one has

$$u = (f \circ g)(u) = f(g(u)) = f(a) = e^{ia}.$$

Furthermore,

$$\sigma(a) \subset [-\pi, \pi],$$

since $\|a\| = \|g(u)\| \leq \|g\|_\infty = \pi$. ■

VI.1.3 Lemma. *If N is a von Neumann algebra, then $K_1^{\mathrm{top}}(N) = \{0\}$.*

Proof. We first check that $\mathrm{GL}_1(N)$ is path connected. Since N is closed under polar decomposition, we can write any element in N as a product of an invertible positive element and a unitary. Any positive invertible element $x \in N$ is connected to the identity by

$$t \mapsto e^{t \log x},$$

with $\log x$ defined by functional calculus. It is thus sufficient to prove that any unitary is connected to $\mathbb{1}$. By lemma VI.1.2 we see that any unitary $u \in N$ can be written as $u = e^{ia}$, for some self-adjoint $a \in N$. Now, we have a norm continuous path

$$t \mapsto e^{ita} \in \mathrm{GL}_1(N)$$

connecting $\mathbb{1}$ and u.
Since $M_n(N)$ is a von Neumann algebra we also see that each $\mathrm{GL}_n(N)$ is path connected. This shows that

$$K_1^{\mathrm{top}}(N) = \varinjlim \mathrm{GL}_n(N) / \mathrm{GL}_n(N)_0 = \{0\}.$$

■

VI.1.4 Definition. A **finite von Neumann algebra** is a von Neumann algebra N together with a function $\tau : N \to \mathbb{C}$ satisfying the following properties:

(i) τ is a positive linear functional,

(ii) τ is faithful, i.e. $\tau(a) = 0$ implies $a = 0$, for all $a \in N_+$,

(iii) τ is tracial, i.e. $\tau(ab) = \tau(ba)$, for all $a, b \in N$,

(iv) τ is normal, i.e. if x_ι is a net in N and $x_\iota \uparrow x$ strongly, then $\tau(x_\iota) \to \tau(x)$.

VI.1.5 Example. Let (X, μ) be a standard probability space. Then $L^\infty(X, \mu)$ is an abelian von Neumann algebra. It is also finite, when equipped with the trace

$$\tau(f) := \int_X f \, d\mu.$$

Another source for finite von Neumann algebras are countable discrete groups. Let Γ be such a group. Then we can represent Γ by the left regular representation on $\ell^2(\Gamma)$ by $\lambda_s(\delta_g) := \delta_{sg}$. Now we define the von Neumann algebra

$$L\Gamma := \{\lambda_s : s \in \Gamma\}''.$$

Then $L\Gamma$ admits a finite trace

$$\tau(x) = \langle x(\delta_e), \delta_e \rangle.$$

We want to emphasize that we are **not** exclusively working with II_1 factors. In particular, our determinant will depend heavily on the trace, with which our von Neumann algebra is endowed.

VI.2 The determinant

VI.2.1 Definition. Consider a finite von Neumann algebra (N, τ). Note that $\tau : N \to \mathbb{C}$ descends to a map on $\mathrm{HC}_0(N)$. Again, one needs that τ is norm continuous and tracial. As before, let

$$\tilde{\tau} := \tau \circ \mathrm{ch}^{\mathrm{rel}} : K_1^{\mathrm{rel}}(N) \to \mathbb{C}.$$

VI.2.2. Using our exact sequence (see lemma III.1.3) together with the fact that $K_1^{\mathrm{top}}(N) = \{0\}$ (see lemma VI.1.3), the comparison sequence simplifies to

$$\begin{array}{ccccccc} K_2^{\mathrm{top}}(N) & \xrightarrow{\partial} & K_1^{\mathrm{rel}}(N) & \xrightarrow{\theta} & K_1^{\mathrm{alg}}(N) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \tilde{\tau}} & & & & \\ & & \mathbb{C} & & & & \end{array}$$

We want to use $\tilde{\tau}$ to define the determinant on $K_1^{\mathrm{alg}}(N)$. Again, we need to see how two lifts of the same element $[g] \in K_1^{\mathrm{alg}}(N)$ are related to each other. Contrary to trace class projections, it turns out that in this case $\tilde{\tau}$ factors through $2\pi i \cdot \mathbb{R}$ on the image of ∂.

VI.2.3 Lemma.

$$\tilde{\tau}(\operatorname{Im}\partial) \subset 2\pi i\ \mathbb{R} = i\ \mathbb{R}.$$

Proof. Copying the proof of lemma V.2.1 we see that

$$\tilde{\tau}(\partial(\gamma_p)) = 2\pi i(\tau \circ \mathrm{TR})(p),$$

where p is a projection in $M_\infty(N)$ and γ_p as in theorem II.6.2. Since TR and τ are positive, we see from theorem II.6.2 again, that

$$\tilde{\tau}(\operatorname{Im}\partial) \subset i\ \mathbb{R}.$$

■

VI.2.4 Remark. The surjectivity of θ shows that each element in the first algebraic K-theory of N can be lifted to some path in the first relative K-theory. If $[\sigma_i]$ are two lifts of the same element, exactness at $K_1^{\mathrm{rel}}(N)$ and lemma VI.2.3 show that $\tilde{\tau}([\sigma_1]) - \tilde{\tau}([\sigma_2]) \in i\mathbb{R}$. The construction and proofs are analogous to those of the Fredholm determinant, as can be seen in theorem V.2.3. We can make the following definition:

VI.2.5 Definition. If $[x] \in K_1^{\mathrm{alg}}(N)$ and $[\sigma] \in K_1^{\mathrm{rel}}(N)$ is a lift of $[x]$, we may define

$$\det_\tau(x) := \tilde{\tau}([\sigma]) + i\mathbb{R}.$$

This gives a well-defined homomorphism

$$\det_\tau : K_1^{\mathrm{alg}}(N) \to \mathbb{C}/i\mathbb{R}.$$

VI.2.6. Define the map $\Re : \mathbb{C}/i\mathbb{R} \to \mathbb{R} : z \mapsto (z + z^*)/2$. To get a real valued determinant, we can postcompose $\det_\tau$ with the map $\exp \circ \Re$. Since the latter map is a group isomorphism, we will view $\det_\tau$ from now on as a map

$$\det_\tau : K_1^{\mathrm{alg}}(N) \to \mathbb{R}_+^*.$$

VI.2.7 Lemma. *Assume $[x] \in K_1^{\mathrm{rel}}(N)$ can be expressed as an exponential, i.e. $x = \exp(a)$. Then*

$$\det([x]) = e^{\Re((\tau \circ \mathrm{TR})(a))}.$$

Proof. Define the path $\sigma : [0, 1] \to \mathrm{GL}(N)$ by

$$\sigma(t) := e^{ta}.$$

Clearly, σ gives rise to an element in $K_1^{\mathrm{rel}}(N)$, which is a lift of $[x]$. By definition, we get

$$\det([x]) = e^{\Re((\tau \circ \mathrm{ch}^{\mathrm{rel}})[\sigma])}$$

But

$$\begin{aligned}\mathrm{ch}^{\mathrm{rel}}([\sigma]) &= \mathrm{TR}\left(\int_0^1 \frac{d\sigma}{dt}\sigma^{-1}\, dt\right) \\ &= \mathrm{TR}\left(\int_0^1 a\, dt\right) = \mathrm{TR}(a).\end{aligned}$$

■

VI.2.8. We now want to show that our determinant coincides with the definition of the Fuglede-Kadison determinant, as originally defined in [3] for II_1-factors. Their formula is

$$\det{}_{\mathrm{FK}}(g) := e^{\tau(\log(|g|))} \qquad (g \in \mathrm{GL}_1(N))$$

where $|g| = \sqrt{g^*g}$. We want to emphasize that our determinant is defined for arbitrary finite von Neumann algebras.

VI.2.9. As for the case of the Fredholm determinant, we will write $\det_\tau(g)$ for an application of the map

$$\mathrm{GL}_1(N) \twoheadrightarrow K_1^{\mathrm{alg}}(N) \xrightarrow{\det_\tau} \mathbb{R}_+^*$$

to g.

VI.2.10 Lemma. *For all $g \in K_1^{\mathrm{rel}}(N)$, we have*

$$\det{}_\tau(g) = \det{}_\tau(|g|).$$

Proof. Using polar decomposition, we find that $g = u\,|g|$. It is thus sufficient to prove that $\det_\tau(u) = 1$. As proved in lemma VI.1.2 we may write

$$u = e^{ia}$$

where $a^* = a$. By lemma VI.2.7 we find that $\det(u)$ is the identity, since $\tau(\mathrm{TR}(ia))$ is purely imaginary. ■

VI.2.11 Proposition. *Let N be a II_1-factor with its unique normalized trace τ. Then, on $\mathrm{GL}_1(N)$ we have*

$$\det{}_\tau = \det{}_{\mathrm{FK}}.$$

Proof. Let $g \in \mathrm{GL}_1(N)$. By the previous lemma and lemma VI.2.7 we see that

$$\det{}_\tau(g) = \det{}_\tau(|g|) = \det{}_\tau(e^{\log|g|}) = e^{\tau(\log|g|)} = \det{}_{\mathrm{FK}}(g).$$

■

VI.2.12 Remark. The definition of $\det_\tau$ immediately gives the same properties of the Fredholm determinant, as summarized in corollary V.2.5.

VII Semi-finite von Neumann algebras

VII.1 Preliminaries

VII.1.1 Definition. Let N be a von Neumann algebra. In the previous section we assumed our trace to be finite. In this section we will drop the assumption of finiteness and a **trace** will be a function

$$\tau : N_+ \to [0, \infty]$$

satisfying the following properties:

1. $\tau(x + y) = \tau(x) + \tau(y)$ for all $x, y \in N_+$,
2. $\tau(tx) = t\tau(x)$, for all $t \in [0, \infty)$ and $x \in N_+$,
3. $\tau(zz^*) = \tau(z^*z)$, for all $z \in N$.

As usual, we define $0 \cdot \infty$ to be 0. A trace is called

- **normal**, if
$$x_\iota \uparrow x \quad \Rightarrow \quad \tau(x_\iota) \uparrow \tau(x),$$
where x_ι is an increasing net that converges strongly to x.
- **faithful**, if for all $x \in N_+$ the following implication holds:
$$\tau(x) = 0 \quad \Rightarrow \quad x = 0.$$
- **semi-finite**, if for all $x \in N_+$ there exists an increasing net x_ι in N_+ such that $x_\iota \uparrow x$ and $\tau(x_\iota) < \infty$. In terms of the next definition, this means that N_τ^+ is strongly dense in N_+, i.e.
$$\overline{N_\tau^+}^{\mathrm{SOT}} = N_+.$$

VII.1.2 Definition. A **semi-finite von Neumann algebra** (N, τ) consists of a von Neumann algebra N together with a normal, faithful and semi-finite trace τ.

VII.1.3. Before we study semi-finite von Neumann algebras in more detail, we recall some basic properties of traces.

VII.1.4 Definition. Given a trace τ on N we define

- $N_\tau^2 := \{x \in N : \tau(x^*x) < \infty\}$

- $N_\tau^+ := \{x \in N_+ : \tau(x) < \infty\}$
- $N_\tau := \mathrm{span}_{\mathbb{C}} N_\tau^+$.

As we will see in the next lemma, the set N_τ can also be written as

$$N_\tau = N_\tau^2 N_\tau^2 = \mathrm{span}_{\mathbb{C}}\{xy : x, y \in N_\tau^2\}.$$

VII.1.5 Lemma. *Let τ be a trace on N. Then τ has the following properties:*

(i) N_τ^2 *is a $*$-ideal inside N.*

(ii) $N_\tau^2 N_\tau^2$ *is a $*$-subalgebra of N and equals N_τ.*

(iii) N_τ *is a $*$-ideal inside N and $(N_\tau)_+ = N_\tau^+$, where $(N_\tau)_+ = N_\tau \cap N_+$.*

(iv) τ *extends uniquely to a positive linear functional $\tau : N_\tau \to \mathbb{C}$ and $\tau(x^*) = \overline{\tau(x)}$ for $x \in N_\tau$.*

(v) The Cauchy-Schwarz inequality holds, i.e.

$$|\tau(y^*x)|^2 \leq \tau(y^*y)\tau(x^*x),$$

whenever $x \in N$ and $y \in N_\tau$.

(vi) $\tau(xy) = \tau(yx)$ *for all $x \in N$ and $y \in N_\tau$ and all $x, y \in N_\tau^2$.*

Proof. These facts are proven in [11, Lemma 5.1.2, Remark 5.2.1, Proposition 5.5.2]. Another very good treatment of traces can be found in [12, Chapter 2]. ■

VII.1.6 Lemma. *Let τ be a trace on N. If $x \in N$, $\tau(|x|) < \infty$, then $x \in N_\tau$ and*

$$|\tau(x)| \leq \tau(|x|).$$

It is furthermore true that

$$|\tau(xy)| \leq \tau(|xy|) \leq \|x\| \, \tau(|y|),$$

where $x \in N$ and $y \in N_\tau$. Similarly

$$|\tau(yx)| \leq \tau(|yx|) \leq \|x\| \, \tau(|y|).$$

See [13, Theorem 8, Chapter 6, p. 118].

VII.2 The trace ideal

VII.2.1 Definition. Let (N, τ) be a semi-finite von Neumann algebra. Define

$$\mathscr{L}^1_\tau(N) := \{x \in N : \tau(|x|) < \infty\}.$$

We call this the **trace ideal** associated to τ. The terminology is justified by the next proposition.

VII.2.2 Proposition. *Let (N, τ) be a semi-finite von Neumann algebra. Then $\mathscr{L}^1_\tau(N)$ is a $*$-ideal inside N and*

$$\|\cdot\|_1 : \mathscr{L}^1_\tau(N) \to [0, \infty) : x \mapsto \tau(|x|)$$

is a norm. Furthermore,

$$\mathscr{L}^1_\tau(N) = N_\tau.$$

Proof. From lemma VII.1.6 we see that $\mathscr{L}^1_\tau(N) \subset N_\tau$. For the reverse implication we first prove that $\mathscr{L}^1_\tau(N)$ is a vector space. Since it is clearly closed under scalar multiplication, we are done if the function $\|\cdot\|_1$ satisfies the triangle inequality. In [13, Corollary 1, Chapter 6, p. 118] it is proven that

$$\tau(|x|) = \sup_{\|y\| \leq 1} |\tau(yx)| \qquad (x \in \mathscr{L}^1_\tau(N)).$$

Applying this to $x + y$ with $x, y \in \mathscr{L}^1_\tau(N)$ the result follows. Since $N^+_\tau \subset \mathscr{L}^1_\tau(N)$ and using that $\mathscr{L}^1_\tau(N)$ is a vector space, we see that

$$N_\tau = \mathrm{span}_{\mathbb{C}} N^+_\tau \subset \mathscr{L}^1_\tau(N).$$

■

VII.2.3 Remark. If (N, τ) is a semi-finite von Neumann algebra, it is clear that we have a normed space

$$(\mathscr{L}^1_\tau(N), \|\cdot\|_1).$$

Note that $\|x\|_1 = 0$ implies that $|x| = 0$ and hence $x^*x = 0$. Applying the C^*-identity to it yields that $\|x\| = 0$ and so $x = 0$.
Now, let

$$\|x\|_\tau := \|x\| + \|x\|_1 .$$

It is then easy to verify that

$$(\mathscr{L}^1_\tau(N), \|\cdot\|_\tau)$$

is a normed algebra. Note that $\|\cdot\|_\tau$ is submultiplicative since

$$\begin{aligned} \|xy\|_\tau &= \|xy\| + \|xy\|_1 \leq \|x\| \|y\| + \|x\| \|y\|_1 \\ &= \|x\| \|y\|_\tau \leq \|x\|_\tau \|y\|_\tau . \end{aligned}$$

VII.2.4 Lemma. *The normed algebra* $(\mathscr{L}^1_\tau(N), \|\cdot\|_\tau)$ *is in fact a Banach algebra (see [14, A4]).*

VII.2.5 Example. The standard example one should have in mind is $(L^\infty(\mathbb{R}), \lambda)$ with λ the Lebesgue measure. This is an abelian von Neumann algebra. Endowing it with the trace

$$\tau(f) := \int_{\mathbb{R}} f \; d\lambda,$$

one obtains a semi-finite von Neumann algebra. Its trace ideal then equals $L^1(\mathbb{R}, \lambda) \cap L^\infty(\mathbb{R}, \lambda)$.

VII.2.6 Lemma. *The pair* $(\mathscr{L}^1_\tau(N), N)$ *is in fact a relative pair of Banach algebras.*

Proof. We verify all properties, as stated in definition II.7.1.

1. We already saw that $(\mathscr{L}^1_\tau(N), \|\cdot\|_\tau)$ is a Banach algebra.
2. Let $a, b \in N$ and $j \in \mathscr{L}^1_\tau(N)$. We need to prove that
$$\|ajb\|_\tau \leq \|a\| \, \|j\|_\tau \, \|b\| .$$
Using lemma VII.1.6 we get
$$\begin{aligned} \|ajb\|_\tau &= \|ajb\| + \|ajb\|_1 \leq \|a\| \, \|j\| \, \|b\| + \|a\| \, \|j\|_1 \, \|b\| \\ &= \|a\| \left(\|j\| + \|j\|_1\right) \|b\| \\ &= \|a\| \, \|j\|_\tau \, \|b\| . \end{aligned}$$
3. Let $j \in \mathscr{L}^1_\tau(N)$. Then
$$\|j\| \leq \|j\| + \|j\|_1 = \|j\|_\tau .$$

■

VII.2.7. The goal of the rest of this section is to prove that the first topological K-theory of the trace ideal of a semi-finite von Neumann algebra is trivial. As for the case of the Fredholm determinant, this will us allow to define a determinant on the first algebraic K-theory of the relative pair $(\mathscr{L}^1_\tau(N), N)$. The next two lemmas and corollaries will be essential for our goal.

VII.2.8 Lemma. *There exist* $c, C > 0$ *such that for all* $x \in [-\pi, \pi]$ *the following holds:*

$$c(1 - \cos x) \leq x^2 \leq C(1 - \cos x).$$

Proof. To prove the claim it is sufficient to look at $[0, \pi]$ since all involved functions are even. Note that the function

$$f : [0, \pi] \to \mathbb{R} : x \mapsto \begin{cases} 2 & \text{if } x = 0 \\ \frac{x^2}{1-\cos x} & \text{if } x \in (0, \pi] \end{cases}$$

is continuous and strictly positive. The only critical point is $x = 0$ but using l'Hopital's rule one can check that

$$\lim_{x \to 0} \frac{x^2}{1 - \cos x} = 2.$$

Since f is a strictly positive continuous function on the compact interval $[0, \pi]$ it follows that f has lower and upper bounds $c, C > 0$, i.e.

$$0 < c \leq f(x) \leq C \qquad (x \in [0, \pi]).$$

Since $1 - \cos(x) \geq 0$ on $[0, \pi]$ we conclude that

$$c(1 - \cos x) \leq x^2 \leq C(1 - \cos x) \qquad (x \in [0, \pi]).$$

■

VII.2.9 Corollary. *Define $K := [-\pi, \pi]$ and let*

$$\Omega_K := \{x \in N : \sigma(x) \subset K\}.$$

Then the following holds:

$$e^{ia} - \mathbb{1} \in \mathscr{L}^1_\tau(N) \quad \iff \quad a \in \mathscr{L}^1_\tau(N) \qquad (a \in \Omega_K).$$

Furthermore, there exists a constant $c > 0$ with

$$\left| e^{ia} - \mathbb{1} \right| \leq c\,|a| \qquad (a \in \Omega_K \cap \mathscr{L}^1_\tau(N)).$$

Proof. Let $a \in \Omega_K$ and define the unitary element $u := e^{ia}$. Define

$$f, g : \sigma(a) \to \mathbb{R}_+$$

by $f(x) = x^2$ and $g(x) = 1 - \cos x$. By lemma VII.2.8 there exist $c, C > 0$ with

$$cg(x) \leq f(x) \leq Cg(x).$$

(I) Assume $u \in \mathscr{L}^1_\tau(N)$. Then, we see that

$$\begin{aligned} |a|^2 &= a^*a = a^2 = f(a) \leq Cg(a) = C(\mathbb{1} - \cos a) \\ &= \frac{C}{2}(e^{-ia} - \mathbb{1})(e^{ia} - \mathbb{1}) = \frac{C}{2}(u^* - \mathbb{1})(u - 1) = \frac{C}{2}|u - \mathbb{1}|^2. \end{aligned}$$

Taking square roots gives

$$|a| \leq \sqrt{\frac{C}{2}}\,|u - \mathbb{1}|.$$

This proves that $a \in \mathscr{L}^1_\tau(N)$.

(II) Now, assume $a \in \mathscr{L}^1_\tau(N)$. Then

$$\left|e^{ia} - \mathbb{1}\right|^2 = 2(\mathbb{1} - \cos(a)) = 2g(a) \leq \frac{2}{c}a^2.$$

Taking square roots gives

$$\left|e^{ia} - \mathbb{1}\right| \leq \sqrt{\frac{2}{c}}\,|a|.$$

This also proves the inequality as stated in the corollary.

■

VII.2.10 Lemma. *Let $I \subset \mathbb{R}$ be some compact interval. Then there exists a $C > 0$ such that*

$$(e^x - 1)^2 \leq Cx^2 \qquad (x \in I).$$

Let $J \subset (0, \infty)$ be some compact interval. Then there exists a $D > 0$ such that

$$(\log x)^2 \leq D(x - 1)^2 \qquad (x \in J).$$

Proof. Assume $0 \in I$. Let

$$f(x) := \begin{cases} \frac{(e^x-1)^2}{x^2} & \text{if } x \in I \setminus \{0\} \\ 1 & \text{if } x = 0. \end{cases}$$

Then $f : I \to \mathbb{R}_+$ is continuous and hence bounded by some $C > 0$. It follows that

$$(e^x - 1)^2 \leq Cx^2 \qquad (x \in I).$$

Notice, that the inequality is also valid for $x = 0$. If $0 \notin I$, the result is immediate.

For the second statement, we can proceed similarly. Define

$$g(x) := \begin{cases} \frac{(\log x)^2}{(x-1)^2} & \text{if } x \in J \setminus \{1\} \\ 1 & \text{if } x = 1. \end{cases}$$

Again, g is continuous on J and hence bounded by some $D > 0$. It follows that

$$(\log x)^2 \leq D(x-1)^2 \qquad (x \in J).$$

As before, if $1 \notin J$, the claim is trivial. ■

VII.2.11 Corollary. *Let $I \subset \mathbb{R}$ and $J \subset (0, \infty)$ be compact intervals. Let $a \in N$ be a self-adjoint element with $\sigma(a) \subset I$ and let $x \in N$ be a positive element with $\sigma(x) \subset J$. As before, we define for each compact subset $K \subset \mathbb{R}$ the set*

$$\Omega_K := \{y \in N : \sigma(y) \subset K\}.$$

Then, there exist $C, D > 0$ with :

$$a \in \mathscr{L}^1_\tau(N) \cap \Omega_I \quad \Rightarrow \quad |e^a - \mathbb{1}| \leq C\,|a|$$

and

$$x \in \mathscr{L}^1_\tau(N) \cap \Omega_J \quad \Rightarrow \quad |\log x| \leq D\,|x - \mathbb{1}|\,.$$

In particular, $e^a - \mathbb{1}$ and $\log x$ are in the trace ideal if a resp. $x - \mathbb{1}$ are.

Proof. This is an immediate consequence of VII.2.10. One proceeds similarly as in the proof of corollary VII.2.9. ■

VII.2.12 Lemma. *Let (N, τ) be a semi-finite von Neumann algebra. Then $U_1(\mathscr{L}^1_\tau(N))$ is path connected, where $U_1(\mathscr{L}^1_\tau(N))$ consists of unitaries $u \in N$ such that $u - \mathbb{1} \in \mathscr{L}^1_\tau(N)$.*

Proof. Let $u \in N$ be some unitary with $u \in \mathscr{L}^1_\tau(N)$. Since u is unitary, we may write

$$u = e^{ia},$$

where a is some self-adjoint $a \in N$ such that $\sigma(a) \subset [-\pi, \pi]$. From corollary VII.2.9 we see that

$$a \in \mathscr{L}^1_\tau(N).$$

With $K = [-\pi, \pi]$ and Ω_K as in VII.2.9, we see that $ta \in \Omega_K \cap \mathscr{L}^1_\tau(N)$, for all $t \in [0, 1]$. Applying the inequality as stated in the same corollary, we see that

$$\left|e^{ita} - \mathbb{1}\right| \leq c\,|ta| = c\;t\,|a|\,. \qquad (\star)$$

It follows, that there is a continuous path

$$t \mapsto e^{ita} \in U_1(\mathscr{L}^1_\tau(N))$$

connecting $\mathbb{1}$ and u. ■

VII.2.13 Remark. Note that the path $\sigma(t) := e^{ita}$ should be continuous with respect to the norm $\|\cdot\|_\tau$. Since the exponent is constructed via the functional calculus on N, continuity of σ is ensured if the path is continuous with respect to $\|\cdot\|_1$. This can be seen by using the inequality we established in $(\star)$ of the previous proof:

$$\left|e^{ita} - \mathbb{1}\right| \leq c\, t\, |a|\,,$$

where $c > 0$ is some constant. For $s, t \in [0,1]$ with $t < s$ know that $s - t \in [0,1]$. Since

$$\begin{aligned}\left|e^{ita} - e^{isa}\right| &= \left|e^{ita}(\mathbb{1} - e^{i(s-t)a})\right| \leq \left\|e^{ita}\right\| \left|\mathbb{1} - e^{i(s-t)a}\right| \\ &= \left|\mathbb{1} - e^{i(s-t)a}\right| \\ &\leq c(s-t)\,|a|\,,\end{aligned}$$

it follows that

$$\left\|e^{ita} - e^{isa}\right\|_1 \leq c(s-t)\,\|a\|_1\,.$$

By interchanging the roles of s and t, one finally gets that

$$\left\|e^{ita} - e^{isa}\right\|_1 \leq c\,|s-t|\,\|a\|_1 \qquad (s, t \in [0,1]).$$

VII.2.14 Lemma. *Let (N, τ) be a semi-finite von Neumann algebra. If $x \in \mathrm{GL}_1(\mathscr{L}^1_\tau(N))$ and $x \in N_+$, there exists a continuous path $\sigma : [0,1] \to \mathrm{GL}_1(\mathscr{L}^1_\tau(N))$ connecting $\mathbb{1}$ and x.*

Proof. Let x be as in the statement. Since $x > 0$ we can take the logarithm of x by means of continuous functional calculus on N. We can now apply corollary VII.2.11 to x and $a := \log x$. Let I and J be compact sets containing $\sigma(a)$ resp. $\sigma(x)$. As in the corollary, take the corresponding constants $C, D > 0$. Then, we see that

$$|a| = |\log x| \leq D\,|x - \mathbb{1}|\,.$$

It follows that a is in the trace ideal. For $t \in [0,1]$ we thus have that $ta \in \mathscr{L}^1_\tau(N) \cap \Omega_I$. This implies

$$\left|e^{t\log x} - \mathbb{1}\right| = \left|e^{ta} - \mathbb{1}\right| \leq D\,|ta| = D\; t\,|a|\,.$$

By the same method as in VII.2.13 we see that

$$t \mapsto e^{t\log x} \in \mathrm{GL}_1(\mathscr{L}^1_\tau(N))$$

is a continuous path connecting $\mathbb{1}$ and x. ■

VII.2.15 Corollary. *Let (N, τ) be a semi-finite von Neumann algebra. Then* $\mathrm{GL}_1(\mathscr{L}^1_\tau(N))$ *is path-connected.*

Proof. Let $g \in \mathrm{GL}_1(\mathscr{L}^1_\tau(N))$. Using polar decomposition write $g = u\,|g|$ with u unitary. Combining lemma VII.2.12 and lemma VII.2.14 we see that there are paths σ_0 and σ_1 with values inside $\mathrm{GL}_1(\mathscr{L}^1_\tau(N))$ connecting $\mathbb{1}$ with u resp. $|g|$. The result follows by considering $\sigma_0\sigma_1$. ■

VII.2.16. The next step is to extend the result of corollary VII.2.15 to general $\mathrm{GL}_n(\mathscr{L}^1_\tau(N))$. Define a trace on $M_n(N)_+$ by

$$\tau_n(x) := \sum_{i=1}^{n} \tau(x_{ii}) = \tau(\mathrm{TR}(x)) \qquad (x \geq 0).$$

It can then be verified that $(M_n(N), \tau_n)$ is a semi-finite von Neumann algebra.

VII.2.17 Lemma. *If (N, τ) is a semi-finite von Neumann algebra, we have*

$$\mathscr{L}^1_{\tau_n}(M_n(N)) = M_n(\mathscr{L}^1_\tau(N)).$$

Furthermore, this identity is an isomorphism of Banach algebras (without requiring it to be an isometry).

Proof. **(I)** Assume $x \in M_n(\mathscr{L}^1_\tau(N))$. Then $\tau(|x_{ij}|) < \infty$ for all i and j. We want to show that $\tau_n(|x|) < \infty$. To this end, we use polar decomposition and write $x = u\,|x|$, with $u, |x| \in M_n(N)$. Write $|x| = u^*x$. Then

$$\tau(|x|_{ii}) = \tau\left(\sum_{k=1}^{n} u^*_{ki} x_{ki}\right) \qquad (\star)$$

But $x_{ki} \in \mathscr{L}^1_\tau(N)$ so each term $u^*_{ki}x_{ki}$ lies in the ideal. We see that the sum is an element of $(N_\tau)_+ = N^+_\tau$. So $\tau(|x|_{ii}) < \infty$ and we see that

$$\tau_n(|x|) = \sum_{i=1}^{n} \tau(|x|_{ii}) < \infty.$$

(II) Now, assume that $x \in \mathscr{L}^1_{\tau_n}(M_n(N))$. For $a \in N$ define the matrix

$$e_{ij}(a)$$

to have a at position (i, j) and 0 everywhere else. We want to show show that each entry of x lies in the trace ideal $\mathscr{L}^1_\tau(N)$. Fix indices i and j. By noting that

$$e_{ij}(x_{ij}) = e_{ii}(\mathbb{1})\; x\; e_{jj}(\mathbb{1}),$$

it follows that $e_{ij}(x_{ij}) \in \mathscr{L}^1_{\tau_n}(M_n(N))$ since this is an ideal. Furthermore,

$$|e_{ij}(x_{ij})|^2 = e_{ji}(x^*_{ij})e_{ij}(x_{ij}) = e_{jj}(|x_{ij}|^2).$$

Taking square roots gives that

$$|e_{ij}(x_{ij})| = e_{jj}(|x_{ij}|).$$

This shows that

$$\tau(|x_{ij}|) = \tau_n(e_{jj}(|x_{ij}|)) = \tau_n(|e_{ij}(x_{ij})|) < \infty.$$

(III) Now, we show that

$$\mathrm{id} : (\mathscr{L}^1_{\tau_n}(M_n(N)), \|\cdot\|_{\tau_n}) \to (M_n(\mathscr{L}^1_\tau(N)), \|\cdot\|_{M_n(\mathscr{L}^1_\tau(N))})$$

is continuous. As usual, we have that

$$\|x\|_{M_n(\mathscr{L}^1_\tau(N))} = \sum_{i,j=1}^n \|x_{ij}\|_\tau = \sum_{i,j=1}^n \|x_{ij}\| + \|x_{ij}\|_1 .$$

Since $\|x_{ij}\| \leq \|x\|$ and $\|x_{ij}\|_1 \leq \|x\|_1$ we see that

$$\|x\|_{M_n(\mathscr{L}^1_\tau(N))} \leq n^2 \|x\|_{\tau_n} .$$

This proves the assertion. From the open mapping theorem it follows that the inverse is also continuous.

■

VII.2.18 Theorem. *Let (N, τ) be a semi-finite von Neumann algebra. Then, the first topologial K-theory of the Banach algebra $(\mathscr{L}^1_\tau(N), \|\cdot\|_\tau)$ is trivial, i.e.*

$$K_1^{\mathrm{top}}(\mathscr{L}^1_\tau(N)) = \{0\}.$$

Proof. Let $n \in \mathbb{N}$. By VII.2.17 it follows that

$$\mathrm{GL}_n(\mathscr{L}^1_\tau(N)) = \mathrm{GL}_1(M_n(\mathscr{L}^1_\tau(N))) \cong \mathrm{GL}_1(\mathscr{L}^1_{\tau_n}(M_n(N))).$$

Here $\cong$ means isomorphic as topological groups, since we have an isomorphism of Banach algebras $M_n(\mathscr{L}^1_\tau(N)) \to \mathscr{L}^1_{\tau_n}(M_n(N))$. By lemma VII.2.12 the right hand side is trivial, which proves the result. ■

VII.3 The determinant

VII.3.1. We are now prepared to define a determinant for semi-finite von Neumann algebras, which is an analogue of the Fredholm determinant.
The next lemma enables us to replace idempotents by projections. Here, the issue is that we are working with the trace ideal instead of the whole algebra N.

VII.3.2 Lemma. *Let* $e \in \mathrm{Idem}(\mathscr{L}^1_\tau(N))$. *Then* e *is similar to a projection* $p \in \mathscr{L}^1_\tau(N)$. *In particular, the similarity is established by an element in* $\mathrm{GL}_1(\mathscr{L}^1_\tau(N))$.

Proof. We need to look back at the proof of lemma II.2.14. There, we constructed the invertible element and the projection as follows: For $a := e^* - e$ we let

$$\begin{aligned} g &:= \mathbb{1} + a^*a \in \mathrm{Inv}(N), \\ p &:= ee^*g^{-1} \in \mathrm{Proj}(N), \\ h &:= \mathbb{1} - p + e \in \mathrm{Inv}(N). \end{aligned}$$

Since $a^*a \in \mathscr{L}^1_\tau(N)$, we see that $g \in \mathrm{GL}_1(\mathscr{L}^1_\tau(N))$. Since $\mathscr{L}^1_\tau(N)$ is an ideal, it is clear that $p \in \mathscr{L}^1_\tau(N)$. To finish the proof, note that $h \in \mathrm{GL}_1(\mathscr{L}^1_\tau(N))$ since $e - p \in \mathscr{L}^1_\tau(N)$. We thus see that our construction works equally well for $\mathscr{L}^1_\tau(N)$. The projection e is conjugated by $h \in \mathrm{GL}_1(\mathscr{L}^1_\tau(N))$ to the projection p in $\mathscr{L}^1_\tau(N)$. ■

VII.3.3 Corollary. *Let* $e \in \mathrm{Idem}_\infty(\mathscr{L}^1_\tau(N))$. *Then* e *is similar, by an element in* $\mathrm{GL}(\mathscr{L}^1_\tau(N))$, *to a projection in* $M_\infty(\mathscr{L}^1_\tau(N))$.

Proof. We may use lemma VII.2.17, which says that the following two $*$-algebras are **equal** (as $*$-algebra)

$$M_n(\mathscr{L}^1_\tau(N)) = \mathscr{L}^1_{\tau_n}(M_n(N)).$$

Since no topological properties are involved, the result follows immediately from the previous lemma. ■

VII.3.4. We are now going to use K-theory of relative pairs to define our determinant, just as we did for the Fredholm determinant. Note that we have proven in lemma VII.2.6 that $(\mathscr{L}^1_\tau(N), N)$ is a relative pair of Banach algebras. A special case of this construction is the Fredholm-determinant as defined for the pair $(\mathscr{L}^1(H), B(H))$, where H is a separable Hilbert space.

VII.3.5 Corollary. *As usual, let* $\tilde{\tau} := \tau \circ \mathrm{ch}^{\mathrm{rel}}$, *defined on* $K_1^{\mathrm{rel}}(\mathscr{L}^1_\tau(N), N)$. *Then*

$$\tilde{\tau}(\mathrm{Im}\,\partial) \subset i\mathbb{R},$$

where

$$\partial : K_2^{\mathrm{top}}(\mathscr{L}_\tau^1(N)) \to K_1^{\mathrm{rel}}(\mathscr{L}_\tau^1(N))$$

is induced by

$$\gamma \mapsto (t \mapsto \gamma(e^{2\pi i t})).$$

Proof. We already have proved that each idempotent is similar to a projection. Furthermore, as proved in [4, Theorem 9.3.1.] one also has Bott periodicity for Banach algebras. This shows that loops in the second topological K-theory of $\mathscr{L}_\tau^1(N)$ are again represented by products of the form $[\gamma_p][\gamma_q]^{-1}$ where p and q are projections and $\gamma_p(z) = zp + \mathbb{1} - p$. Now, the proof for the Fredholm determinant may be copied. Note that we do not get $2\pi i\mathbb{Z}$ as values, since projections in von Neumann algebras do not necessarily have a natural number as trace. In particular, one uses the formula

$$\tilde{\tau}(\partial([\gamma_p])) = 2\pi i\ (\tau \circ \mathrm{TR})(p) \in 2\pi i\mathbb{R} = i\mathbb{R}.$$

■

VII.3.6 Definition. Combining everything, we can define

$$\det_\tau : K_1^{\mathrm{alg}}(\mathscr{L}_\tau^1(N), N) \to \mathbb{C}/(i\mathbb{R}) \cong \mathbb{R}_+^*$$

by imposing the following diagram to commute:

$$\begin{array}{ccccccc} K_2^{\mathrm{top}}(\mathscr{L}_\tau^1(N)) & \xrightarrow{\partial} & K_1^{\mathrm{rel}}(\mathscr{L}_\tau^1(N), N) & \xrightarrow{\theta} & K_1^{\mathrm{alg}}(\mathscr{L}_\tau^1(N), N) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \tilde{\tau}\circ\partial} & & \downarrow{\scriptstyle \tilde{\tau}} & & \downarrow{\scriptstyle \det_\tau} & & \\ i\mathbb{R} & \rightarrowtail & \mathbb{C} & \twoheadrightarrow & \mathbb{C}/(i\mathbb{R}) & & \end{array}$$

The proof that this is possible is exactly the same as for the Fredholm determinant. Recall that it is essential to know that $\mathscr{L}_\tau^1(N)$ has trivial first topological K-theory.

VII.3.7. To be a bit more explicit about $\det_\tau$: If g is invertible in N and $g - \mathbb{1}$ lies in the trace ideal, one may compute

$$\det_\tau(g) = \exp\left(\int_0^1 \tau(\sigma'(t)\sigma(t)^{-1})\ dt\right),$$

where σ is a path connecting $\mathbb{1}$ and x such that $\sigma(t)$ is invertible in N and $\sigma(t) - \mathbb{1}$ is in $\mathscr{L}_\tau^1(N)$, for all $t \in [0, 1]$.

VII.3.8. The next proposition shows that $\det_\tau$ can be computed by the same formula as the Fuglede-Kadison determinant.

VII.3.9 Proposition. *Let* (N, τ) *be a semi-finite von Neumann algebra. For* $g \in \mathrm{GL}_1(\mathscr{L}^1_\tau(N))$*, the following holds:* $\log |g|$ *sits in the trace ideal and*

$$\det_\tau(g) = e^{\tau(\log|g|)} = \exp(\tau(\log(g^*g)^{\frac{1}{2}})).$$

Proof. Let $g \in \mathrm{GL}_1(\mathscr{L}^1_\tau(N))$. Since

$$g^*g - \mathbb{1} = g^*(g-1) + (g - \mathbb{1})^*,$$

we see that $g^*g - \mathbb{1} \in \mathscr{L}^1_\tau(N)$. Define $x := g^*g$. Then $x > 0$ and $x - \mathbb{1}$ is in the trace ideal. By corollary VII.2.11 we see that $\log x$ and hence also $\frac{1}{2}\log x$ lies in the trace ideal. The same corollary shows that

$$e^{\frac{1}{2}\log x} \in \mathrm{GL}_1(\mathscr{L}^1_\tau(N)).$$

But $e^{\frac{1}{2}\log x} = |g|$, which proves the first assertion.
Now, write $g = u\,|g|$. Then

$$(u - \mathbb{1})^* = -u^*(g - \mathbb{1}) + (|g| - \mathbb{1}).$$

It follows that $u \in \mathrm{GL}_1(\mathscr{L}^1_\tau(N))$. In particular, we may compute

$$\det_\tau(g) = \det_\tau(u)\det_\tau(|g|).$$

As usual, write $u = e^{ia}$. Then

$$\sigma(t) = e^{ita}$$

is a path inside $\mathrm{GL}_1(\mathscr{L}^1_\tau(N))$ connecting $\mathbb{1}$ and u (cf. lemma VII.2.12). Then

$$\det_\tau(u) = e^{\Re(\tau(ia)))},$$

since the relative Chern character of σ is

$$\int_0^1 \sigma'\sigma^{-1}\,dt = ia.$$

But $e^{\Re(\tau(ia))} = e^0 = 1$, since the real part of $\tau(ia)$ is trivial (a is self-adjoint). We see that

$$\det_\tau(g) = \det_\tau(|g|).$$

As shown in lemma VII.2.14, we see that

$$\omega(t) := e^{t\log|g|}$$

is a continuous path inside $\mathrm{GL}_1(\mathscr{L}^1_\tau(N))$ connecting $\mathbb{1}$ and $|g|$. The relative Chern character of ω is

$$\int_0^1 \omega'\omega^{-1}\ dt = \log|g|\,.$$

Since $\log|g|$ is self-adjoint we see that $\tau(\log|g|)$ is a real number. This gives

$$\det{}_\tau(g) = \det{}_\tau(|g|) = e^{\tau(\log|g|)}.$$

■

Appendix

A Continuous paths into $\mathrm{GL}(A)$

A.1.1. This appendix is meant to prove a theorem which is somehow ancient folklore. It is a well-known result, that paths into $\mathrm{GL}(A)$ are homotopic to ones taking values in some $\mathrm{GL}_n(A)$ for a fixed $n \in \mathbb{N}$. However, we want to make the effort to prove this rather technical result.

A.1.2. To be more precise: Let A be a unital Banach algebra. The main purpose of this appendix is to prove that a continuous path

$$\sigma : [0,1] \to \mathrm{GL}(A), \quad \sigma(0) = \mathbb{1}$$

is homotopic to a smooth path σ', which is an element of $R(A)_1$ as defined in II.4.1. This means that there is a homotopy with fixed endpoints between σ and σ' such that σ' has the following properties:

1. σ' is smooth.
2. There exists a fixed $n \in \mathbb{N}$ such that σ' takes values in $\mathrm{GL}_n(A)$.
3. σ' is constant in a neighborhood of 0 and 1.

The next example, due to Jens Kaad, shows that there are paths $\sigma : [0,1] \to \mathrm{GL}(A)$ which do not factor through some $\mathrm{GL}_n(A)$.

A.1.3 Example. For each $n \in \mathbb{N}$ we make continuous functions $f_n : [0,1] \to [0,1]$ with $\|f\|_\infty = 1$, $f(0) = 0$ and $f(t) = 0$ for $t > \frac{1}{n}$, as in figure 1. Now, we may define a path $\sigma : [0,1] \to \mathrm{GL}(A)$ by

$$\sigma(t) := \mathbb{1} + \sum_{n=1}^{\infty} 2^{-n} f_n(t) \cdot \mathbb{1}_n.$$

First, note that for each $t > 0$, the sum is finite. Fix some $t \in (0,1]$ and $k \in \mathbb{N}$ with $k^{-1} < t$. Then

$$\sigma(t) = \mathbb{1} + \sum_{n=1}^{k} 2^{-n} f_n(t) \cdot \mathbb{1}_n.$$

Then, we have

$$\|\sigma(t) - \mathbb{1}\| \leq \sum_{n=1}^{k} 2^{-n} < 1 \qquad (t \in [0,1]).$$

In particular, $\sigma(t) \in \mathrm{GL}_n(A)$ for all $t \in [0,1]$.

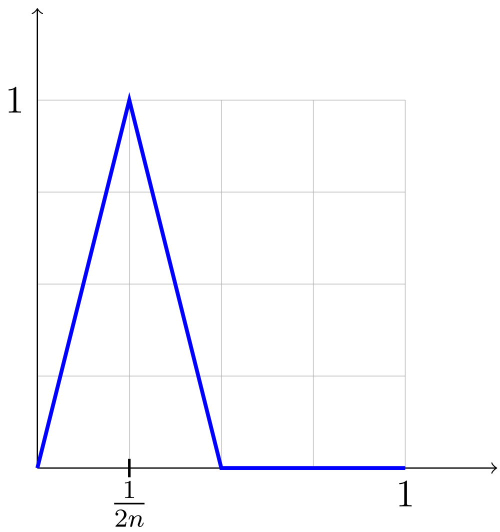

Figure 1: Graph of f_n

A.1.4 Definition. Let $n \in \mathbb{N}$ and $\delta := \frac{1}{n}$. Define $t_i := i\delta$ and let $V_i = V_i(\delta) := (t_i - \delta, t_i + \delta) \cap [0,1]$. By $\chi^\delta = \{\chi_i^\delta\}_{i=0}^n$ we mean a partition of unity subordinate to the open cover $\{V_i\}_{i=0}^n$, where χ^δ has the following properties:

1. The points 0 and 1 lie **only** in the support of χ_0 resp. χ_n.
2. For each i, χ_i has constant value 1 on $(t_i - \delta/3, t_i + \delta/3)$.

See figures 2, 3 and 4.

A.1.5 Definition. Fix $\delta = \frac{1}{n}$. If $\gamma : [0,1] \to \mathrm{GL}(A)$ is a continuous path

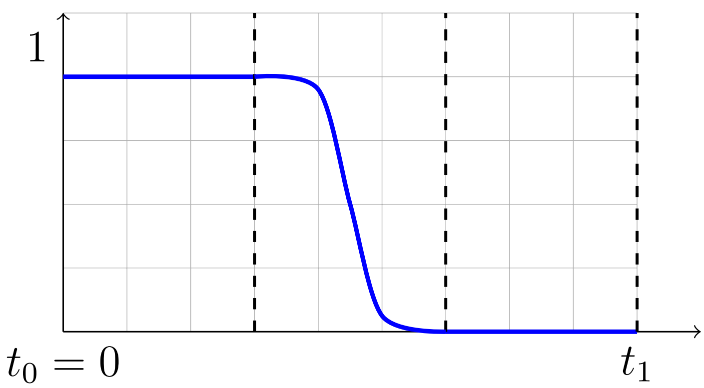

Figure 2: Construction of χ_0

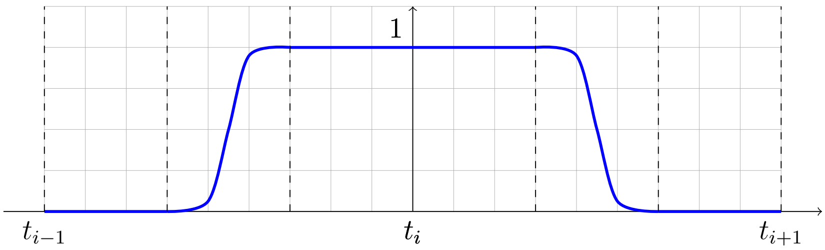

Figure 3: Construction of χ_i for $0 < i < n$

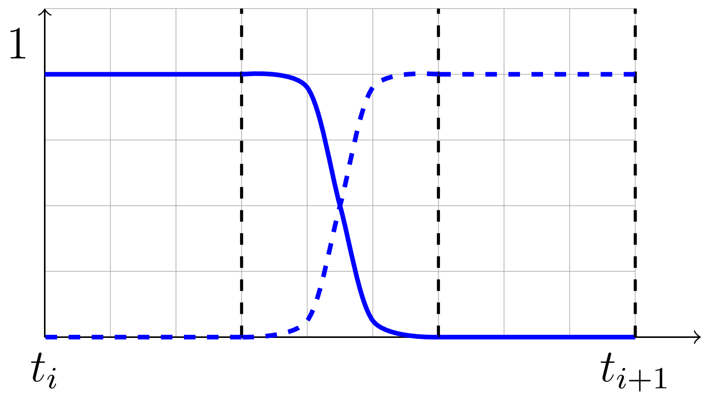

Figure 4: The overlap $V_i \cap V_{i+1}$

starting at $\mathbb{1}$ such that

$$\forall i \in \{0, 1, \cdots, n\} \; \forall t \in V_i : \left\|\mathbb{1} - \gamma(t)^{-1}\gamma(t_i)\right\| < 1,$$

we say that γ is δ-compatible.

A.1.6 Definition. Let $H : [0,1] \times [0,1] \to \mathrm{GL}(A)$ be a homotopy and let $\delta = \frac{1}{n}$. Just as we did for paths, we say that H is δ-compatible, if

$$\forall i, j \in \{0, 1, 2, \cdots, n\} \; \forall (t, s) \in V_i \times V_j : \left\|\mathbb{1} - H(t, s)^{-1}H(t_i, s_j)\right\| < 1,$$

where $t_i = s_i = i\delta$.

A.1.7 Lemma. *Let γ be as before. Then there exists some $n \in \mathbb{N}$ such that γ is δ-compatible, where $\delta = \frac{1}{n}$.*

Proof. Define a function

$$\Omega : [0,1] \times [0,1] \to \mathrm{GL}(A) : (t, s) \mapsto \gamma(s)^{-1}\gamma(t).$$

Then, as GL(A) is a topological group, Ω is continuous and hence by compactness of $[0,1] \times [0,1]$ uniformly continuous. It follows that there exists some $\alpha > 0$ such that

$$|s - s'| + |t - t'| < \alpha \quad \Rightarrow \quad \|\Omega(t, s) - \Omega(t', s')\| < 1.$$

Now, let $n \in \mathbb{N}$ such that $\frac{1}{n} < \alpha$. We claim that γ is δ-compatible, with $\delta := \frac{1}{n}$. If $|t - t_i| < \delta$, i.e. $t \in V_i$, it follows that

$$\left\|\mathbb{1} - \gamma(t)^{-1}\gamma(t_i)\right\| = \|\Omega(t_i, t_i) - \Omega(t_i, t)\| < 1.$$

■

A.1.8 Lemma. *Let $H : [0,1] \times [0,1] \to \mathrm{GL}(A)$ be a homotopy. Then, there exists a $\delta = \frac{1}{n}$ such that H is δ-compatible.*

Proof. Having proven lemma A.1.7, it is clear how to proceed: Define a function

$$\Omega : [0,1]^4 \to \mathrm{GL}(A) : (t, s, p, q) \mapsto H(p, q)^{-1}H(t, s).$$

This function is again continuous and hence uniformly continuous. It follows that there exists a $n \in \mathbb{N}$ such that, with $\delta = \frac{1}{n}$, we have

$$|t - t'| + |s - s'| + |p - p'| + |q - q'| < 2\delta \Rightarrow \|\Omega(t, s, p, q) - \Omega(t', s', p', q')\| < 1.$$

Let $t_i = s_i = i\delta$. If now $|t - t_i| < \delta$ and $|s - s_j| < \delta$, we see

$$\begin{aligned}
&\|\Omega(t_i, s_j, t_i, s_j) - \Omega(t_i, s_j, t, s)\| \\
&= \left\|H(t_i, s_j)^{-1} H(t_i, s_j) - H(t, s)^{-1} H(t_i, s_j)\right\| \\
&= \left\|\mathbb{1} - H(t, s)^{-1} H(t_i, s_j)\right\| < 1.
\end{aligned}$$

■

A.1.9 Proposition. *Let γ be a continuous path starting at $\mathbb{1}$. Then there exists a smooth path γ' starting at $\mathbb{1}$, which takes values in a fixed $\mathrm{GL}_n(A)$ such that γ and γ' are homotopic.*

Proof. We prove the proposition in steps.

(I) By lemma A.1.7, there exists a p.o.u. χ^δ such that γ is δ-compatible. Now, we define homotopies

$$H_i : V_i \times [0, 1] \to M_\infty(A)$$

by

$$H_i(t, s) := \gamma(t_i)\psi(s) + (1 - \psi(s))\gamma(t),$$

where

$$\psi : [0, 1] \to [0, 1]$$

is a smooth function, with $\psi(0) = 0$ and $\psi(1) = 1$, with the further requirement that $\psi(s) = 1$, for s in an open neighborhood of 1.

(II) To get the final homotopy, we use our partition of unity to glue these locally defined H_i together. Let

$$H : [0, 1] \times [0, 1] \to \mathrm{GL}(A) : (t, s) \mapsto \sum_{i=0}^{n} \chi_i(t) H_i(t, s).$$

We need to check that this is well defined, i.e. that H lands in $\mathrm{GL}(A)$. Let $(t, s) \in [0, 1] \times [0, 1]$. Then

$$\begin{aligned}
\left\|\mathbb{1} - \gamma(t)^{-1} H(t, s)\right\| &= \left\|\mathbb{1} - \gamma(t)^{-1} \sum_i \chi_i(t)(\gamma(t_i)\psi(s) + (1 - \psi(s))\gamma(t))\right\| \\
&= \left\|\sum_i \chi_i(t)(\mathbb{1} - \gamma(t)^{-1}\gamma(t_i)\psi(s) - (1 - \psi(s))\mathbb{1}\right\| \\
&\le \sum_i \chi_i(t) \left\|\psi(s)\mathbb{1} - \psi(s)\gamma(t)^{-1}\gamma(t_i)\right\| \\
&\le \sum_i \chi_i(t) \left\|\mathbb{1} - \gamma(t)^{-1}\gamma(t_i)\right\| \\
&< \sum_i \chi_i(t) = 1.
\end{aligned}$$

To get the last inequality, note that $\|\mathbb{1} - \gamma(t)^{-1}\gamma(t_i)\| < 1$, for all $t \in V_i$. This is true, as γ is δ-compatible.

It follows that $\gamma(t)^{-1}H(t,s)$ is invertible, and hence also $H(t,s)$.

(III) Now, the main work is done. By construction, we have $H(-,0) = \gamma$. Furthermore, we have that

$$H(-,1) = \sum_i \chi_i H_i(-,1) = \sum_i \chi_i \gamma(t_i),$$

where $H_i(-,1)$ is just the constant path taking value $\gamma(t_i)$. Since there is a $k \in \mathbb{N}$ with $\gamma(t_i) \in \mathrm{GL}_k(A)$ for all $i = 0, 1, \cdots, n$, we see that

$$\gamma' := H(-,1) : [0,1] \to \mathrm{GL}_k(A)$$

is a smooth path homotopic to γ, with $\gamma'(0) = \mathbb{1}$.

(IV) Since H is clearly smooth, it remains to prove that H leaves endpoints fixed.

$$\begin{aligned}\forall s \in [0,1] : H(0,s) &= \sum_i \chi_i(0) H_i(0,s) = \chi_0(0) H_0(0,s)\\ &= H_0(0,s) = \gamma(t_0) = \gamma(0) = \mathbb{1}.\end{aligned}$$

Here, we make use of the fact that $0 = t_0$ is just contained in the support of χ_0. By the same argument, we get

$$\begin{aligned}\forall s \in [0,1] : H(1,s) &= \sum_i \chi_i(1) H_i(1,s) = \chi_n(1) H_n(1,s)\\ &= H_n(1,s) = \gamma(t_n) = \gamma(1).\end{aligned}$$

■

A.1.10 Corollary. *If $\gamma : [0,1] \to \mathrm{GL}_m(A)$ is a smooth path starting at $\mathbb{1}$ which is δ-compatible (with $\delta = \frac{1}{n}$, for some $n \in \mathbb{N}$), then there exists a smooth homotopy $H : [0,1] \times [0,1] \to \mathrm{GL}_k(A)$ with fixed endpoints, which makes γ and*

$$\sum_{i=0}^{n} \chi_i^\delta \gamma(t_i)$$

homotopic. We can furthermore ensure that

$$\forall s \in (1 - \delta/3, 1] : H(-,s) = H(-,1) = \sum_i \chi_i^\delta \gamma(t_i).$$

Proof. Just do the same proof (starting at **(II)**) as for proposition A.1.9. The resulting homotopy H is exactly what we need here. Smoothness of γ and ψ ensures that all H_i are smooth and so is H. Furthermore, in the same proof, the usage of ψ gives that the $H_i(-, s)$ are constant, with value $\gamma(t_i)$, for $s \in (1 - \delta/3, 1]$. ∎

A.1.11 Lemma. *Let $\gamma_1, \gamma_2 : [0,1] \to \mathrm{GL}_n(A)$ be smooth paths starting at $\mathbb{1}$. Assume H is a homotopy $[0,1] \times [0,1] \to \mathrm{GL}(A)$ with fixed endpoints between γ_1 and γ_2. Then, there exists a homotopy H' with fixed endpoints between γ_1 and γ_2, which is smooth and takes values in some $\mathrm{GL}_m(A)$.*

Proof. **(I)** First, we modify H. By lemma A.1.8, there exists some $\delta = \frac{1}{n}$, such that H is δ-compatible. This means that

$$\left\| \mathbb{1} - H(t,s)^{-1} H(t_i, s_j) \right\| < 1,$$

whenever $(t,s) \in V_{ij} = V_i \times V_j$. Furthermore, we get a partition of unity subordinate to the open cover $\{V_{ij}\}_{i,j=0}^n$ by $\chi_{ij}^\delta = \chi_i^\delta \cdot \chi_j^\delta$.

Now, we can define $\tilde{H} : [0,1] \times [0,1] \to \mathrm{GL}(A)$ by

$$\tilde{H}(t,s) := \sum_{i,j} \chi_{ij}(t,s) H(t_i, s_j).$$

We need to verify that $\tilde{H}$ really lands in $\mathrm{GL}(A)$. Let $(t,s) \in [0,1]^2$. Then

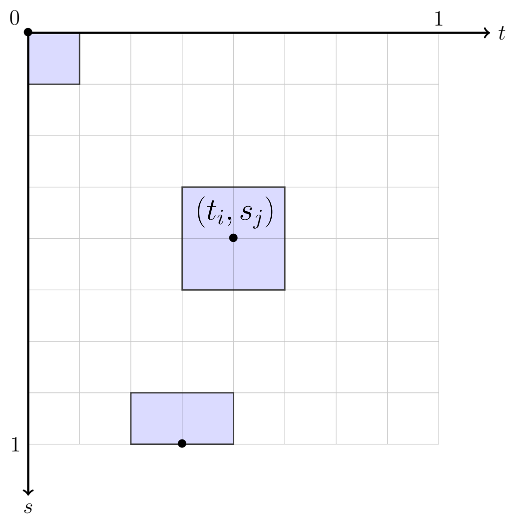

Figure 5: Construction of $\tilde{H}$

$$\begin{aligned}\left\|\mathbb{1} - H(t,s)^{-1}\tilde{H}(t,s)\right\| &= \left\|\sum_{i,j}\chi_{ij}(t,s)\mathbb{1} - \sum_{i,j}\chi_{ij}(t,s)H(t,s)^{-1}H(t_i,s_j)\right\| \\ &= \left\|\sum_{i,j}\chi_{ij}(t,s)(\mathbb{1} - H(t,s)^{-1}H(t_i,s_j))\right\| \\ &\leq \sum_{i,j}\chi_{ij}(t,s)\left\|\mathbb{1} - H(t,s)^{-1}H(t_i,s_j)\right\| \\ &< \sum_{i,j}\chi_{ij}(t,s) = 1.\end{aligned}$$

Note that we use that $\|\mathbb{1} - H(t,s)^{-1}H(t_i,s_j)\| < 1$ if $(t,s) \in V_{ij}$.

(II) Using that $\chi_j(0) = \delta_{j,0}$, we see that

$$\begin{aligned}\tilde{H}(t,0) &= \sum_{i,j}\chi_i(t)\chi_j(0)H(t_i,s_j) = \sum_i \chi_i(t)H(t_i,s_0) \\ &= \sum_i \chi_i(t)H(t_i,0) = \sum_i \chi_i(t)\gamma_0(t_i).\end{aligned}$$

By construction, we also know that $\gamma_0 = H(-,0)$ is δ-compatible. Since γ_0 is smooth and takes values in $\mathrm{GL}_n(A)$, we may apply corollary A.1.10 to conclude that there exists a smooth homotopy

$$H_0 : [0,1] \times [0,1] \to \mathrm{GL}_m(A)$$

with fixed endpoints between γ_0 and $\sum_i \chi_i\gamma_0(t_i) = \tilde{H}(-,0)$. By construction of the χ_i and the corollary, we now also have that

$$\forall s \in [0,\delta/3) : H_0(-,1-s) = H_0(-,1) = \tilde{H}(-,0) = \tilde{H}(-,s).$$

Since this two homotopies are constant on the overlap, we can concatenate H_0 and $\tilde{H}$ in a smooth way, to get a homotopy

$$H_0 * \tilde{H}$$

between γ_0 and $\tilde{H}(-,1)$. As usual, the $*$ operation is defined as

$$(H_0 * \tilde{H})(t,s) = (H_0(-,t) * \tilde{H}(-,t))(s),$$

where two paths, say α and β, are concatenated by

$$(\alpha * \beta)(s) = \begin{cases}\alpha(2s) & \text{if } s \in [0,1/2] \\ \beta(2s-1) & \text{if } s \in (1/2,1].\end{cases}$$

(III) Applying the same trick, we also can make a smooth homotopy H_1 taking values in some $\mathrm{GL}_{m'}(A)$ between $\tilde{H}(-,1)$ and γ_1 such that the resulting homotopy
$$H_0 * \tilde{H} * H_1$$
is smooth and takes values in some $\mathrm{GL}_p(A)$, with $p \in \mathbb{N}$.

■

We can now collect our findings in a theorem:

A.1.12 Theorem. *Let* $\gamma : [0,1] \to \mathrm{GL}(A)$ *be a continuous path such that* $\gamma(0) = \mathbb{1}$. *Then there exists a smooth path* $\gamma' : [0,1] \to \mathrm{GL}_n(A)$ *such that* γ *and* γ' *are homotopic via a smooth homotopy* H*, which also takes values in some* $\mathrm{GL}_m(A)$. *Furthermore* γ' *is constant in a neighborhood of* 0 *and* 1. *In addition to that, we can also ensure that* $H(t,-)$ *and* $H(-,s)$ *are of the same type as* γ'*, for all* s *and* t.
In particular, this result justifies definition II.4.1.

A.1.13 Lemma. *Theorem A.1.12 is also true for non-unital Banach algebras.*

Proof. We apply our construction to $\mathrm{GL}(A^{\sim})$. The essence of the proof is then to note the following: If γ is a continuous path into $\mathrm{GL}(A) \subset \mathrm{GL}(A^{\sim})$ with $\gamma(0) = \mathbb{1}$ we obtain the homotopic path by considering a convex combination of some values $\gamma(t_i)$, such that their convex sum is an invertible element. It is thus sufficient to remark the following: If $g_1, \cdots, g_n \in \mathrm{GL}(A)$ and
$$\sum_{i=1}^{n} \theta_i g_i \in \mathrm{GL}(A^{\sim}),$$
then also
$$\sum_{i=1}^{n} \theta_i g_i \in \mathrm{GL}(A),$$
where $\theta_i \geq 0$ and $\theta_1 + \cdots + \theta_n = 1$.
To see this, note that
$$\left(\sum_{i=1}^{n} \theta_i g_i\right) - \mathbb{1} = \sum_{i=1}^{n} \theta_i (g_i - \mathbb{1}) \in M_\infty(A).$$

■

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