Rokhlin actions and permanence properties

1 Preliminaries

These notes follow mainly the ideas of [Izu04] and [BS16].

1.1 Conventions and Notation. If not stated otherwise, G will denote a **finite** group. By "$\alpha\colon G \curvearrowright A$", we mean that $\alpha\colon G \to \mathrm{Aut}(A)$ is an action of G on a C*-algebra A and $\alpha(g)$ will be written as α_g for $g \in G$. If A is a C*-algebra, the sequence algebra of A will be denoted by

$$A_\infty := \ell^\infty(A)/c_0(A).$$

The central sequence algebra of A will be denoted by $A_\infty \cap A'$.

1.2 Definition. Let A be a unital C*-algebra. We say that $\alpha\colon G \curvearrowright A$ has the *Rokhlin property* if there exist projections $(e_g)_{g\in G}$ in $A_\infty \cap A'$ such that

(*i*) $\alpha_g^\infty(e_h) = e_{gh}$ for all $g, h \in G$.

(*ii*) $1_{A_\infty} = \sum_{g\in G} e_g$.

We also say that the $(e_g)_{g\in G}$ are Rokhlin projections for the action α.

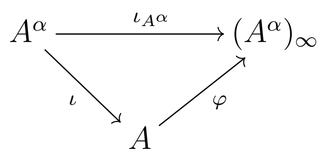
1.3 Definition. Given an action $\alpha\colon G \curvearrowright A$, we define the conditional expectation

$$E\colon A \to A^\alpha : a \mapsto \frac{1}{|G|}\sum_{g\in G} \alpha_g(a).$$

1.4 Lemma. *Let A be unital and assume $\alpha\colon G \curvearrowright A$ has the Rokhlin property. Then, there exists a $*$-homomorphism*

$$\varphi\colon A \to \big(A^\alpha\big)_\infty,$$

such that $\varphi(a) = a$ for all $a \in A^\alpha$, i.e. such that the following diagram commutes:



Proof. Define

$$\psi\colon A \to A_\infty : a \mapsto \sum_{g\in G} \alpha_g(a)e_g.$$

Clearly, ψ is a $*$-hom. and furthermore $\psi(a) \in (A_\infty)^{\alpha_\infty}$. To finish the proof, note that

$$\phi\colon (A_\infty)^{\alpha_\infty} \to (A^\alpha)_\infty : \big[(x_n)_{\mathbb{N}}\big] \mapsto \big[(E(x_n))_{\mathbb{N}}\big]$$

is a well-defined $*$-isomorphism, which acts as the identity on elements of $(A^\alpha)_\infty$. Therefore,

$$\varphi\colon A \to (A^\alpha)_\infty : a \mapsto \phi(\psi(a))$$

is as required. □

1.5 Lemma. *Assume $\alpha\colon G \curvearrowright A$ has the Rokhlin property. Then*

$$\begin{aligned} &\operatorname{Im}\big(\mathrm{K}_i(\iota)\colon \mathrm{K}_i(A^\alpha) \to \mathrm{K}_i(A)\big) \\ &= \{x \in \mathrm{K}_i(A) : \mathrm{K}_i(\alpha_g)(x) = x,\ \forall g \in G\}, \end{aligned}$$

where $i \in \{0, 1\}$.

Proof. We only have to deal with the case $i = 0$. The case $i = 1$ can then be handled by looking at $A \otimes \mathcal{O}^\infty$, where $\mathcal{O}^\infty$ is a UCT Kirchberg algebra with $\mathrm{K}_*(\mathcal{O}^\infty) = 0 \oplus \mathbb{Z}$.
Let φ and ψ be as in Lemma 1.4and fix some projection $p \in A$ such that $\alpha_g(p) \sim_{\mathrm{u}} p$ for all $g \in G$. Let $u_g \in A$ be unitaries such that

$$u_g^*\alpha_g(p)u_g = p \qquad (g \in G).$$

Let $(e_g)_{g\in G}$ be Rokhlin projections in $A_\infty \cap A'$. Define the unitary

$$w := \sum_{g\in G} u_g e_g \in A_\infty.$$

Then $wpw^* \in (A_\infty)^{\alpha_\infty}$ since

$$\begin{aligned} wpw^* &= \sum_{g,h\in G} u_g e_g p e_h^* u_h^* = \sum_{h\in G} u_h p u_h^* e_h = \sum_{h \in G} \alpha_h(p) e_h \\ &= \psi(p) \in (A_\infty)^{\alpha_\infty}. \end{aligned}$$

It follows that there exists a unitary $u \in A$ with

$$\|\alpha_g(upu^*) - upu^*\| < \frac{1}{4} \qquad (g \in G)$$

Let $a := E(upu^*)$. Then, by the previous calculation

$$\|a - upu^*\| < \frac{1}{4}$$

and $a \in A^\alpha$. Note that

$$\|a - \varphi(upu^*)\| = \|\varphi(a) - \varphi(upu^*)\| < \frac{1}{4}.$$

Since $\varphi(upu^*)$ is a projection in $(A^\alpha)_\infty$ it follows that there exists a projection $q \in A^\alpha$ such that $\|q - a\| < \frac{2}{4}$. Then

$$\begin{aligned} \|q - upu^*\| &\le \|q - a\| + \|a - upu^*\| \\ &< \frac{2}{4} + \frac{1}{4} < 1. \end{aligned}$$

In particular, $upu^* \sim_{\mathrm{u}} q$ and

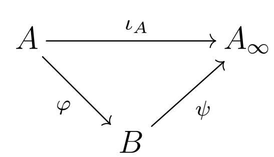
$$K_0(\iota)\big([q]_0\big) = [q]_0 = [upu^*]_0 = [p]_0.$$

□

2 Sequentially split ∗-homomorphisms

The following relies on Barlak and Szabó 's article [BS16].

2.1 Definition. Let $\varphi\colon A \to B$ be a ∗-hom. We say that φ is **sequentially split**, if there exists a ∗-hom. $\psi\colon B \to A_\infty$ such that the following diagram commutes:



If A and B are unital, we require all maps to be unital.

2.2 Example. By Lemma 1.4, the inclusion $\iota\colon A^\alpha \hookrightarrow A$ is sequentially split, if $\alpha\colon G \curvearrowright A$ has the Rokhlin property.

2.3 Example. Let A and B be Kirchberg algebras in the UCT class. Then, there exists a sequentially split $*$-hom. from A to B if and only if $\mathrm{K}_*(A)$ is a pure subgroup of $\mathrm{K}_*(B)$.

2.4 Lemma. *Assume $\varphi\colon A \to B$ is sequentially split and A, B, φ are unital. Then, the following holds:*

(I) $\mathrm{K}_i(\varphi)\colon \mathrm{K}_i(A) \to \mathrm{K}_i(B)$ *is injective for* $i \in \{0,1\}$.

(II) *If B has stable rank one, then A has stable rank one.*

(III) *If B is* AF, *then A is* AF.

(IV) $\varphi_*\colon \mathrm{T}(B) \to \mathrm{T}(A) : \tau \mapsto \tau \circ \varphi$ *is surjective.*

Proof. Assume $\psi\colon B \to A_\infty$ is a unital $*$-hom. such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\quad \iota_A \quad} & A_\infty \\ & {\scriptstyle\varphi}\searrow \qquad \nearrow{\scriptstyle\psi} & \\ & B & \end{array}$$

(**I**) We first prove the case $i = 0$. Look at the induced diagram on K-theory:

$$\begin{array}{ccc} \mathrm{K}_0(A) & \xrightarrow{\quad \mathrm{K}_0(\iota_A) \quad} & \mathrm{K}_0(A_\infty) \\ & {\scriptstyle\mathrm{K}_0(\varphi)}\searrow \qquad \nearrow{\scriptstyle\mathrm{K}_0(\psi)} & \\ & \mathrm{K}_0(B) & \end{array}$$

We just have to show that the map $\mathrm{K}_0(\iota_A)$ is injective. Assume $\mathrm{K}_0(\iota)([p]_0 - [q]_0) = 0$. We may assume that p and q are projections in A and that there exists a unitary $u \in A_\infty$ with $upu^* = q$. Writing $u = \big[(u_n)_\mathbb{N}\big]$ with u_n unitary for all $n \in \mathbb{N}$. Then for large n we see that $\|u_n p u_n^* - q\| < 1$ and hence $p \sim_u q$ in A. So $[p]_0 - [q]_0 = 0$.

The case $i = 1$ can be moved to the case $i = 0$ by taking suspensions. Since the Bott isomorphism is natural the following diagram commutes:

$$\begin{array}{ccc} \mathrm{K}_0(SA) & \xrightarrow{\mathrm{K}_0(S\varphi)} & \mathrm{K}_0(SB) \\ \downarrow & & \downarrow \\ \mathrm{K}_1(A) & \xrightarrow{\mathrm{K}_1(\varphi)} & \mathrm{K}_1(B) \end{array}$$

But $S\varphi$ is sequentially split since φ is and therefore we conclude that $\mathrm{K}_1(\varphi)$ is injective.

(**II**) Let $a \in A$ and $\varepsilon > 0$. Find $g \in B$ invertible, such that $\|\varphi(a) - b\| < \varepsilon$. Then, $\|a - \psi(b)\| = \|\psi(\varphi(a)) - \psi(b)\| < \varepsilon$. Since $\psi(b) = [(x_n)_{\mathbb{N}}]$ is invertible in A_∞, it follows that x_n is invertible, for large n. As

$$\varepsilon > \|a - \psi(b)\| = \limsup_{n \to \infty} \|a - x_n\| \,,$$

we get $\|a - x_n\| < \varepsilon$ for large $n \in \mathbb{N}$ and so a can be approximated by invertibles in A.

(**III**) Let $\mathscr{F} \subseteq A$ be a finite set and $\varepsilon > 0$. Since B is AF, we can find a finite dimensional C*-subalgebra $F_0 \subseteq B$ such that $\varphi(\mathscr{F}) \subseteq_\varepsilon F_0$. Then, $\iota_A(\mathscr{F}) = \psi(\varphi(\mathscr{F})) \subseteq_\varepsilon \psi(F_0)$. With $F := \psi(F_0)$ we have

$$\mathscr{F} \subset_\varepsilon F \text{ in } A_\infty.$$

Since F is finite dimensional, it is weakly semiprojective. Therefore, the inclusion of F into A_∞ can be lifted to $\ell^\infty(A)$:

$$\begin{array}{ccc} & & \ell^\infty(A) \\ & \overset{\phi}{\nearrow} & \downarrow \pi \\ F & \longrightarrow & A_\infty \end{array}$$

Let ϕ_n denote the components of ϕ. Let $x \in \mathscr{F}$. Then there exists $x' \in F$ with $\|x - x'\| < \varepsilon$. Since $x' = \left[(\phi_n(x'))_{\mathbb{N}}\right]$ it follows that

$$\|x - \phi_n(x')\| < \varepsilon$$

for large $n \in \mathbb{N}$. Hence there exists some $n \in \mathbb{N}$ with

$$\mathscr{F} \subseteq_\varepsilon \phi_n(F).$$

(**IV**) Let $\tau \in \mathrm{T}(A)$ and fix a free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$. Define $\tau_\omega \in \mathrm{T}(A_\infty)$ by

$$\tau_\omega\big([(a_n)_{\mathbb{N}}]\big) := \lim_{\omega} \tau(a_n).$$

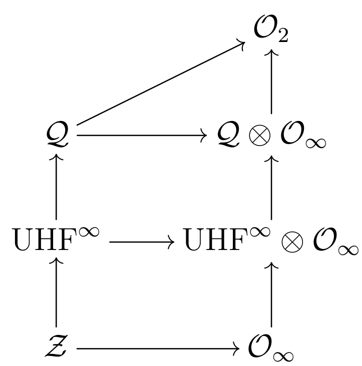
Then, we have that $\varphi_*(\tau_\omega \circ \psi) = \tau_\omega \circ \psi \circ \varphi = \tau_\omega \circ \iota_A = \tau.$ $\square$

3 Strongly self-absorbing algebras

3.1 Definition. Let D be a separable unital C*-algebra. We say that D is strongly self-absorbing if there exists an isomorphism $\varphi\colon D \to D \otimes D$ such that $\varphi \approx_{\text{a.u.}} \mathrm{id} \otimes 1_D$.

3.2 Remark. An immediate consequence of the above definition is that if D is strongly self-absorbing, then D has an approximate inner flip. Thus D is simple and nuclear.

3.3 Example. An exhausting list of all so far known strongly self-absorbing C*-algebras is given by:



3.4 Theorem. *Let A and D be separable* C*-*algebras, where D is strongly self-absorbing and A unital. Then, the following are equivalent:*

(i) *A is D-stable, i.e. $A \otimes D \cong A$,*

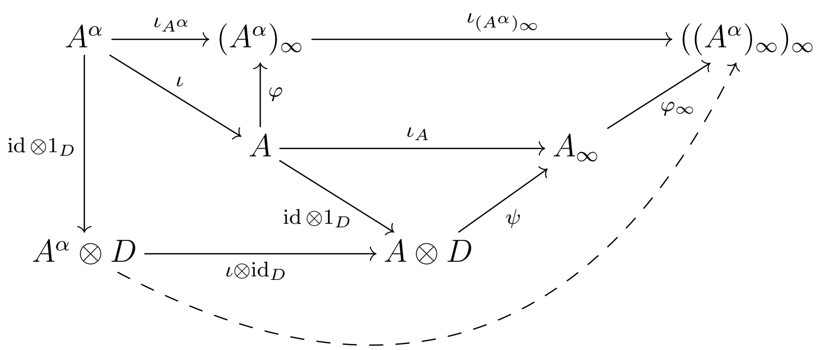
(ii) *D embeds unitally into $A_\infty \cap A'$,*

(iii) *The first factor ebmedding $A \hookrightarrow A \otimes D : a \mapsto a \otimes 1_D$ is sequentially split*

Proof. $(i) \Leftrightarrow (ii)$: This is Theorem 2.2. in [TW07]. For $(i) \Leftrightarrow (iii)$ one uses Theorem 2.3 in [TW07] together with the non-trivial fact that all strongly self-absorbing C*-algebras are K_1-injective. See [Win11]. □

3.5 Corollary. *Assume $\alpha\colon G \curvearrowright A$ has the Rokhlin property. If A is D-stable, where D is a strongly self-absorbing* C*-*algebra, then also A^α is D-stable.*

Proof.



Using [BS16, Lemma 2.2] it follows that $\mathrm{id} \otimes 1_D \colon A^\alpha \to A^\alpha \otimes D$ is sequentially split and therefore A^α is D-stable by Theorem 3.4. □

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