On Arveson's extension theorem

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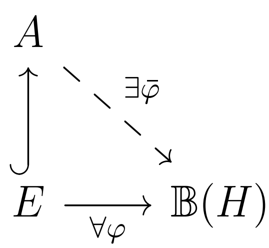
This note should point out, how Arveson's extension theorem for operator systems can be extended to the setting of C*-algebras, with not necessarily unital inclusions. In the following, $A^\sim$ will always denote the forced unitization of a C*-algebra A. So, if A is already unital, then $A^\sim \cong A \oplus \mathbb{C}$.

The following is known as **Arveson's extension theorem**:

Theorem 1: ([BO08, Theorem 1.6.1]) Let A be a unital C*-algebra and $E \subset A$ be an operator subsystem (in particular, $E \subset A$ unitally). Then every c.p.c. map $\varphi\colon E \to \mathbb{B}(H)$ has a c.p.c. extension $\bar{\varphi}$ to A:

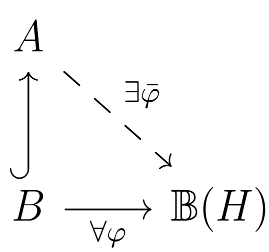


We now try to generalize Theorem 1. First, we need the following Lemma, which is proven in [BO08, Prop. 2.2.1].

Lemma 1: Let $\varphi\colon A \to B$ be a c.p.c. map, where B is unital and A is non-unital. Then φ has a u.c.p. extension $\varphi^\sim$, given by

$$\varphi^\sim\colon A^\sim \to B : a + \lambda 1_{A^\sim} \mapsto \varphi(a) + \lambda 1_B.$$

Theorem 2: Let A be a C*-algebra an $B \subset A$ be C*-subalgebra. Then every c.p.c. map $\varphi\colon B \to \mathbb{B}(H)$ has a c.p.c. extension $\bar{\varphi}$ to A:



Proof. We treat the two cases of A being unital or not separately. Furthermore, we can assume that B is non-unital since otherwise the extension problem is easily proved, by extending φ to a c.p.c. map $\bar{\varphi}$ on $1_B A 1_B$ and then to A by composing with the obvious conditional expectation $E\colon A \to 1_B A 1_B$. So let us assume from now on that B is non-unital.
(I): Assume A is unital. If $1_A \in B$, we are done by Theorem 1. Otherwise, $1_A \notin B$. In that case $B^\sim = B + \mathbb{C}1_A$, where the sum is direct. By Lemma 1 we may extend φ to a u.c.p. map $\varphi^\sim\colon B^\sim \to \mathbb{B}(H)$, by defining $\varphi^\sim(b + \lambda 1_A) := \varphi(b) + \lambda 1_{\mathbb{B}(H)}$. Furthermore, $B^\sim \subset A$ unitally. So by Theorem 1 we have the following diagram

$$\begin{array}{ccc} A & & \\ \uparrow & \searrow^{\overline{\varphi^\sim}} & \\ B^\sim & \xrightarrow[\varphi^\sim]{} & \mathbb{B}(H) \end{array}$$

Clearly, $\overline{\varphi^\sim}$ extends φ.
(II): Assume A is non-unital. Then $B^\sim \subset A^\sim$ unitally, by $b + \lambda 1_{B^\sim} \mapsto b + \lambda 1_{A^\sim}$. Again, using Lemma 1, extend φ to u.c.p. map $\varphi^\sim\colon B^\sim \to \mathbb{B}(H)$. By Theorem 1 we have the following diagram

$$\begin{array}{ccc} A^\sim & & \\ \uparrow & \searrow^{\overline{\varphi^\sim}} & \\ B^\sim & \xrightarrow[\varphi^\sim]{} & \mathbb{B}(H) \end{array}$$

Clearly, $\overline{\varphi^\sim}$ extends φ. $\square$

Bibliography

[BO08] Nathanial P. Brown and Narutaka Ozawa. *C^*-algebras and finite-dimensional approximations*, volume 88 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.