On certain strongly self-absorbing C*-algebras

1 UHF algebras

1.1 Definition. We fix an enumeration $(p_i)_{i=1}^{\infty}$ of the prime numbers, including 1. A **supernatural number** is a sequence $\mathbf{n} = (n_i)_{i=1}^{\infty}$ where $n_i \in \mathbb{N} \cup \{\infty\}$. More suggestively we will denote this number by

$$\mathbf{n} = \prod_{i=1}^{\infty} p_i^{n_i}.$$

For $\mathbf{n}, \mathbf{m}$ supernatural numbers, we define

$$\mathbf{n} \mid \mathbf{m} \qquad \Longleftrightarrow \qquad \forall i \in \mathbb{N} : n_i \leq m_i.$$

We say $\mathbf{n}$ is of infinite type, if

$$n_i > 0 \qquad \Rightarrow \qquad n_i = \infty.$$

1.2 Definition. Let n be a supernatural number. We then define

$$\mathbb{Q}(\mathbf{n}) := \left\{ \frac{x}{y} : x \in \mathbb{Z}, y \in \mathbb{N}, y \mid \mathbf{n} \right\}.$$

1.3 Lemma. *If H is a subgroup of $\mathbb{Q}$ with $1 \in H$, then $H = \mathbb{Q}(\mathbf{n})$ for some supernatural number $\mathbf{n}$.*

1.4 Lemma. *Assume $f : \mathbb{Q}(\mathbf{n}) \to \mathbb{Q}(\mathbf{m})$ is a group homomorphism with $f(1) \neq 0$. Then $\mathbf{n} \mid \mathbf{m}$, if $\mathbf{n}$ is of infinite type.*

Proof. We may assume that $f(1) > 0$. Then, there are $k, l \in \mathbb{N}$ with $f(l) = k$. Now consider the composition

$$g : \mathbb{Q}(nl) \xrightarrow{\cdot l} Q(n) \xrightarrow{f} \mathbb{Q}(m) \xrightarrow{\cdot k^{-1}} \mathbb{Q}(mk)$$

Clearly $g(1) = 1$ and therefore, $\mathbf{n} \cdot l \mid \mathbf{m} \cdot k$. Since $\mathbf{n}$ is of infinite type, this clearly implies that $\mathbf{m}$ is of infinite type and $\mathbf{n} \mid \mathbf{m}$. $\square$

1.5 Definition. A UHF-algebra is a unital countable inductive limit of simple finite dimensional C*-algebras.

1.6 Remark. Any UHF-algebra is simple and has a unique tracial state.

1.7 Theorem. *For every supernatural number* $\mathbf{n}$*, there exists (up to isomorphism) a unique* UHF*-algebra, denoted by* $M_{\mathbf{n}}$*, such that*

$$(\mathrm{K}_0(M_{\mathbf{n}}), [1]_0) \cong (\mathbb{Q}(\mathbf{n}), 1).$$

If

$$\mathbf{n} = \prod_{i=1}^{\infty} p_i^{n_i},$$

then one can realize $M_{\mathbf{n}}$ *as*

$$\lim_{k\to\infty} (M_{r_k}, \varphi_k), \quad \textit{with} \quad r_k := \prod_{i=1}^{k} p_i^{\min(n_i,k)}.$$

The connecting maps φ_k *are given by* $\mathrm{id}_{M_{r_k}} \otimes 1_{s_k}$*, where* $r_k s_k = r_{k+1}$*.*

1.8 Definition. If $\mathbf{n}$ is of infinite type, we say that $M_{\mathbf{n}}$ is a UHF-algebra of infinite type. We denote the class of those algebras by UHF^∞.

1.9 Remark. If $U \in \mathrm{UHF}^\infty$, then U is strongly self-absorbing.

1.10 Example. Two frequently used UHF-algebras of infinite type are:

1. M_{2^∞}, called the CAR-algebra,
2. $Q := M_{\mathbf{n}}$, where $\mathbf{n} = \prod_{i=1}^{\infty} p_i^{\infty}$, called the **universal** UHF-algebra.

The algebra Q is uniquely determined by the fact that $\mathrm{K}_0(Q) = \mathbb{Q}$ and furthermore, Q contains a unital copy of every finite dimensional matrix algebra M_n.

2 Strongly self-absorbing C*-algebras containing a non-trivial projection

2.1 Lemma. *Let* A *be a unital* C**-algebra and let* $n \in \mathbb{N}$*. Assume*

$$n \mid [1_A] \quad \textit{in} \quad V(A),$$

meaning that there exists some element $x \in V(A)$ *with* $nx = [1_A]_{V(A)}$*. Then* M_n *embeds unitally into* A*.*

Proof. Write $x = [p]$, where $p \in M_k(A)$ is a projection. Then, after identifying $M_{kn}(A) \cong M_k(A) \otimes M_n$ it follows that there is a partial isometry $v \in M_k(A) \otimes M_n$ such that

$$vv^* = p \otimes 1_n, \qquad v^*v = \mathrm{diag}(1_A, 0, \ldots, 0) \otimes e_{11}.$$

Let us abbreviate $v^*v = 1_A \otimes e_{11}$ in the following. We define a $*$-homomorphism

$$\psi : M_n \to M_k(A) \otimes M_n : x \mapsto v^*(p \otimes x)v.$$

Indeed, ψ is multiplicative since

$$\begin{aligned} \psi(x)\psi(y) &= v^*(p \otimes x)vv^*(p \otimes y)v \\ &= v^*(p \otimes x)(p \otimes 1_n)(p \otimes y)v \\ &= v^*(p \otimes xy)v = \psi(xy). \end{aligned}$$

Furthermore, the image of ψ lands in the corner

$$(1_A \otimes e_{11})(M_k(A) \otimes M_n)(1_A \otimes e_{11}) \cong A.$$

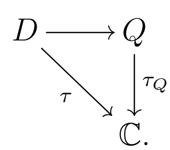
and $\psi(1_n)$ is mapped to the identity of A under this identification. □

2.2 Theorem. *[DRr09, Prop. 2.7] Let D be a strongly self-absorbing quasidiagonal C^*-algebra containing a non-trivial projection. Assume furthermore, that $\mathrm{K}_0(D)$ is torsion-free. Then, there exists some $U \in \mathrm{UHF}^\infty$ such that*

$$(\mathrm{K}_0(D), [1_D]_0) \cong (\mathrm{K}_0(U), [1_U]_0)$$

and $D \otimes U \cong D$.

Proof. Since D is quasidiagonal, $D \otimes Q \cong Q$ and D embeds unitally into Q. Let $\tau : D \to \mathbb{C}$ be the composition making the following diagram commute:



We consider $\tau_* : \mathrm{K}_0(D) \to \mathbb{Q}$ and let $H := \mathrm{Ker}(\tau_*)$. This gives a short exact sequence

$$0 \to H \to \mathrm{K}_0(D) \to \mathrm{Im}(\tau_*) \to 0.$$

Since $\mathbb{Q}$ is torsion-free the functor $- \otimes_{\mathbb{Z}} \mathbb{Q}$ is exact. This produces another short exact sequence:

$$0 \to H \otimes \mathbb{Q} \to \mathrm{K}_0(D) \otimes \mathbb{Q} \xrightarrow{\phi} \mathrm{Im}(\tau_*) \otimes \mathbb{Q} \to 0, \tag{$*$}$$

where $\phi = \tau_* \otimes \mathrm{id}_{\mathbb{Q}}$. Applying the Künneth theorem to Q and D it follows that

$$\mathrm{K}_0(D) \otimes \mathbb{Q} \cong \mathrm{K}_0(D \otimes Q) \cong \mathrm{K}_0(Q) \cong \mathbb{Q}.$$

Since $\mathrm{Im}(\tau_*)$ is a subgroup of $\mathbb{Q}$ containing 1, it follows that ϕ is (after identifications) a non-trivial group homomorphism from $\mathbb{Q}$ to $\mathbb{Q}$. Therefore, ϕ is an isomorphism. By exactness of $(*)$ we get that

$$H \otimes \mathbb{Q} = 0.$$

It follows that H must be a torsion group and, as subgroup of $\mathrm{K}_0(D)$, trivial. Then, $\tau_* : \mathrm{K}_0(D) \to \mathrm{Im}(\tau_*)$ is a group isomorphism. By Lemma 1.3, it follows that

$$\mathrm{Im}(\tau_*) \cong \mathbb{Q}(\mathbf{n}),$$

where $\mathbf{n}$ is a supernatural number of infinite type. Since D contains a non-trivial projection, D contains projections of arbitrary small trace. Therefore, $\mathbb{Q}(\mathbf{n}) \neq \mathbb{Z}$ and hence $(\mathrm{K}_0(D), [1_D]_0) \cong (\mathrm{K}_0(U), [1_U]_0)$ for some $U \in \mathrm{UHF}^\infty$. It remains to show that U embeds unitally into D. It is sufficient to show that M_k embeds unitally into D, whenever $k \mid \mathbf{n}$. Let us fix some k like this. Then $k^{-1} \in \mathbb{Q}(\mathbf{n}) \cong \mathrm{K}_0(D)$. It follows that there exists some $x \in \mathrm{K}_0(D)$ with $kx = [1_D]_0$. Since D is stably finite and $\mathcal{Z}$-stable, $V(D)$ has cancellation (see [Rr04]) and therefore

$$k \mid [1_D] \quad \text{in } V(D).$$

By Lemma 2.1, it follows that M_k embeds unitally into D. □

2.3 Corollary. *Assume D and E are strongly self-absorbing quasidiagonal* C*-*algebras which contain both a non-trivial projection and have torsion-free* K_0-*group. If there are embeddings*

$$\phi : D \hookrightarrow E, \qquad \psi : E \hookrightarrow D,$$

then D and E are U-stable for the same $U \in \mathrm{UHF}^\infty$ and

$$(\mathrm{K}_0(D), [1_D]_0) \cong (\mathrm{K}_0(U), [1_U]_0) \cong (\mathrm{K}_0(E), [1_E]_0).$$

Proof. Combine Theorem 2.2 and Lemma 1.4. □

Bibliography

[DRr09] Marius Dadarlat and Mikael Rø rdam. Strongly self-absorbing C^*-algebras which contain a nontrivial projection. *Münster J. Math.*, 2:35–44, 2009.

[Rr04] Mikael Rø rdam. The stable and the real rank of $\mathscr{Z}$-absorbing C^*-algebras. *Internat. J. Math.*, 15(10):1065–1084, 2004.