Homotopy invariance of quasidiagonality

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Abstract

I wrote these notes as a guideline for my talk at the **kleines seminar** in Münster. The aim is to explain the concept of quasidiagonality for C*-algebras and why this property is invariant under homotopy. This remarkable result has been proven by Voiculescu in [Voi91].

1 Quasicentral approximate units

In this section we will present some preliminaries on quasicentral approximate units. See [Arv77]. By $A, B, C, \dots$ we will denote C*-algebras if not stated otherwise.

1.1 Definition. We say that $h \in A$ is **strictly positive** if and only if $\varphi(h) > 0$ for every $\varphi \in S(A)$, where $S(A)$ denotes the set of states on A.

1.2 Lemma. *The following are equivalent for $h \in A_+$ with $\|h\| \leq 1$:*

1. *h is strictly positive*
2. *$A_h = \overline{hAh} = A$, i.e. the hereditary* C**-algebra generated by h is A*
3. *$\{f_n(h)\}_{n=1}^\infty$ is an approximate unit for A, where*

$$f_n(t) = \begin{cases} 0 & \text{if } t \in [0, 2^{-n}] \\ \text{linear} & \text{if } t \in [2^{-n}, 2^{-n+1}] \\ 1 & \text{if } t > 2^{-n+1} \end{cases}.$$

Proof. See [Bla06, II.4.2.1] . □

1.3 Corollary. *If A is separable, then A has a countable approximate unit $\{u_n\}$ such that $u_{n+1}u_n = u_n$.*

Proof. If $\{e_n\}$ is some countable approximate unit for A then the element

$$h := \sum_n 2^{-n} e_n$$

is strictly positive. Now, we may apply Lemma 1.2 and note that $f_{n+1}f_n = f_n$. □

1.4 Definition. Let K be a closed ideal inside A. If $\{u_\lambda\}$ is an approximate unit for K we say that $\{u_\lambda\}$ is **quasicentral** if

$$\|u_\lambda a - a u_\lambda\| \to 0 \qquad (a \in A).$$

1.5 Theorem. *Let K be a closed ideal in A. Let $\{e_\lambda\}$ be an approximate unit for K. Then there exists a quasicentral approximate unit $\{u_\mu\}$ for K such that for all μ:*

$$u_\mu \in \mathrm{co}(\{e_\lambda\}),$$

where co *denotes the convex hull.*

Proof. See Theorem 1 in [Arv77]. The main ingredient is a separation theorem (Hahn-Banach). □

1.6 Corollary. *Let K be a separable closed ideal inside A, where A is also assumed to be separable. Then K has a countable quasicentral approximate unit $\{u_n\}_{n=1}^{\infty}$ such that $u_{n+1}u_n = u_n$.*

Proof. The proof goes by inductively choosing the u_n. There are two important remarks:

- If $\{e_n\}_{n=1}^{\infty}$ is a countable approximate unit for K, then for all $N \in \mathbb{N}$: $\{e_n\}_{n\geq N}$ is still an approximate unit for K.
- The u_n will be convex combinations of the e_n (which satisfy $e_{n+1}e_n = e_n$). To ensure that $u_{n+1}u_n = u_n$, we throw (after having constructed u_n) away all the e_k which have been used before to build $u_1, \cdots, u_n$. The main tool here is Theorem 1.5.

First, fix some dense countable subsets $\{k_1, k_2, \cdots\}$ and $\{a_1, a_2, \cdots\}$ in K resp. A. Furthermore, since K is separable we can fix a countable approximate unit $\{e_n\}$ for K such that $e_{n+1}e_n = e_n$ (see Corollary 1.3). The first step is now to choose some $u_1 \in \mathrm{co}(\{e_n\}_{n=1}^{\infty})$ with

$$\left[\begin{array}{l} \|k_1 u_1 - k_1\| < 2^{-1}, \\ \|a_1 u_1 - u_1 a_1\| < 2^{-1} \end{array}\right.$$

Having chosen $\{u_1, \cdots, u_n\}$ we construct u_{n+1} in the following way: Pick a large enough $N \in \mathbb{N}$ such that

$$u_1, u_2, \cdots, u_n \in \mathrm{co}(\{e_1, e_2, \cdots, e_{N-1}\}).$$

Then, let $u_{n+1} \in \mathrm{co}(\{e_n\}_{n=N}^{\infty})$ such that

$$\left[\begin{array}{l} \|k_i u_{n+1} - k_i\| < 2^{-(n+1)} \text{ for } i \leq n+1, \\ \|a_i u_{n+1} - u_{n+1} a_i\| < 2^{-(n+1)} \text{ for } i \leq n+1 \end{array}\right.$$

One can now check that $u_{n+1}u_n = u_n$ for all $n \in \mathbb{N}$ and that $\{u_n\}_{n=1}^{\infty}$ is a quasicentral approximate unit for K. □

1.7 Corollary. *Let H be a separable Hilbert space and let $Q \in B(H)$ be a finite rank projection. Then, the statement of Corollary 1.6 holds with $K = K(H)$ and the additional property that each u_n dominates Q. Furthermore, we may assume that each u_n is a finite rank operator.*

Proof. We may choose a countable orthonormal base $\{\psi_n\}$ for H such that the first n_0 basis vectors constitute a basis for the image of Q. Now, we let $\{e_n\}_{n\geq n_0}$ be the approximate unit for $K(H)$ which consists of the finite rank projections e_n which project onto the span of the first n basis vectors. They satisfy $e_{n+1}e_n = e_n$ and all the e_n $(n \geq n_0)$ dominate Q (hence also convex combinations of them). If we now make our quasicentral approximate unit $\{u_n\}$ as in Corollary 1.6 we see that the u_n dominate Q and that each u_n is a finite rank operator. □

2 Quasidiagonality

We first introduce some terminology. A detailed survey can be found in [Bro00]. We will begin with a quite abstract definition of quasidiagonality. However, there are other (equivalent) definitions which might be more intuitive at a first glance.

2.1 Definition. We say that A is **quasidiagonal** (QD) if and only if for every $\epsilon > 0$ and finite subset $\mathfrak{F} \subset A$, there exists some $n \in \mathbb{N}$ and a completely positive contractive (cpc) map $\varphi : A \to M_n(\mathbb{C})$ such that for all $a, b \in \mathfrak{F}$:

$$\left[\begin{array}{l} \|\varphi(ab) - \varphi(a)\varphi(b)\| < \epsilon, \\ \|\varphi(a)\| \geq \|a\| - \epsilon \end{array}\right.$$

This means that φ is almost isometric and multiplicative on $\mathfrak{F}$.

2.2 Remark. Indeed, in the above definition we may assume that $\mathfrak{F} \subset A_{\leq 1}$. In the unital case we may also assume that φ is unital (ucp). See [BO, Lemma 7.1.4].

2.3 Lemma. *Let A be a unital* C*-*algebra. Then A is QD if and only if for every $\epsilon > 0$ and $\mathfrak{F} \subset A_{\leq 1}$ finite there exists a representation $\pi : A \to B(H)$ and a finite rank projection $P \in B(H)$ such that for all $a \in \mathfrak{F}$:*

$$\left[\begin{array}{l} \|\pi(a)P - P\pi(a)\| < \epsilon, \\ \|P\pi(a)P\| \geq \|a\| - \epsilon \end{array}\right.$$

Proof. $\Longleftarrow$) Let $\mathfrak{F} \subset A_{\leq 1}$ finite and $\epsilon > 0$. Assume that such a representation exists. Define the cpc map $\varphi : A \to PB(H)P$ by $\varphi(a) := P\pi(a)P$. Then φ is a cpc map. We check that φ has the desired properties: By definition, we have that

$$\|\varphi(a)\| \geq \|a\| - \epsilon \qquad (a \in \mathfrak{F}).$$

On the other hand, for $a, b \in \mathfrak{F}$ we get that

$$\begin{aligned} \|\varphi(ab) - \varphi(a)\varphi(b)\| &= \|P\pi(a)\pi(b)P - P\pi(a)P\pi(b)P\| \\ &\leq \|P\pi(a)\| \left\|\pi(b)P^2 - P\pi(b)P\right\| \\ &\leq \|[\pi(b), P]\| \\ &< \epsilon. \end{aligned}$$

$\Longrightarrow$) Assume that A is QD and let $\mathfrak{F} \subset A_{\leq 1}$ be finite and $\epsilon > 0$. Since A is QD there exists a ucp map $\varphi : A \to M_n(\mathbb{C})$ such that φ is almost isometric and multiplicative on the elements of $\mathfrak{F} \cup \mathfrak{F}^*$. By Stinespring's Dialation theorem, there exists a unital representation $\pi : A \to B(H)$ such that

$$\varphi(a) = P\pi(a)P \qquad (a \in A),$$

where P is a finite rank projection (identify the corner $PB(H)P$ with $M_n(\mathbb{C})$ again). First, note that for $a \in \mathfrak{F}$:

$$\begin{aligned}\epsilon > \|\varphi(a)\varphi(a^*) - \varphi(aa^*)\| &= \|P\pi(a)P\pi(a^*)P - P\pi(a)\pi(a^*)P\| \\ &= \|(P\pi(a)(1-P))(P\pi(a)(1-P))^*\| \\ &= \|P\pi(a)(1-P)\|^2 .\end{aligned}$$

On the other hand, we have for all $a \in \mathfrak{F}$:

$$\begin{aligned}\|P\pi(a) - \pi(a)P\| & \left\|P\pi(a)P^\perp - P^\perp\pi(a)P\right\| \\ &\overset{(*)}{\leq} \max(\left\|P\pi(a)P^\perp\right\|, \left\|P\pi(a^*)P^\perp\right\|) \\ &= \max(\|\varphi(a)\varphi(a^*) - \varphi(aa^*)\|, \|\varphi(a^*)\varphi(a) - \varphi(a^*a)\|) \\ &< \epsilon.\end{aligned}$$

□

2.4 Remark. To get the inequality in $(*)$, one has to note that the following is true in every C*-algebra A: If $p \in A$ is a projection, then for all $a \in A$ the following holds:

$$\left\|pap^\perp + p^\perp ap\right\| \leq \max(\left\|pap^\perp\right\|, \left\|p^\perp ap\right\|) = \max(\left\|pap^\perp\right\|, \left\|pa^*p^\perp\right\|).$$

To see this, note that

$$\left\|pap^\perp + p^\perp ap\right\|^2 = \left\|\underbrace{p^\perp a^*pap^\perp}_{x} + \underbrace{pa^*p^\perp ap}_{y}\right\|^2.$$

Since x and y are self-adjoint perpendicular elements they generate a commutative C*-algebra. But for functions f, g on a locally compact space we know that $fg = 0$ implies that $\|f + g\|_\infty \leq \max(\|f\|_\infty, \|g\|_\infty)$. Therefore, we get $\|x + y\| \leq \max(\|x\|, \|y\|)$. However,

$$\|x\| = \left\|pap^\perp\right\|^2$$

and

$$\|y\| = \left\|p^\perp ap\right\|^2 = \left\|pa^*p^\perp\right\|^2.$$

2.5 Definition. Let H be a Hilbert space and $\Omega \subset B(H)$. We call Ω a **quasidiagonal set of operators** if for each finite set $\mathfrak{F} \subset \Omega$, each finite set $\chi \subset H$ and $\epsilon > 0$ there exists a finite rank projection $P \in B(H)$ with

$$\left[\begin{array}{ll}\|Pa - aP\| < \epsilon & (a \in \Omega), \\ \|Px - x\| < \epsilon & (x \in \chi)\end{array}\right.$$

2.6 Remark. In fact, the previous definition implies that one may assume $Px = x$ for all $x \in \chi$. See [BO, Prop. 7.2.3].

2.7 Lemma. *Assume $\Omega \subset B(H)$ is a quasidiagonal set of operators and let $\mathfrak{F} \subset \Omega$ be a finite subset. Let $F \in B(H)$ be a finite rank operator such that $0 \leq F \leq 1$ and $\|Fa - aF\| < \epsilon$ for all $a \in \mathfrak{F}$. Then, there exists a finite rank projection $P \in B(H)$ which dominates F and also commutes up to ϵ with $\mathfrak{F}$.*

Proof. Since Ω is a quasidiagonal set of operators we know that there exists a finite rank projection $P \in B(H)$ such that $\mathrm{Im}(F) \subset \mathrm{Im}(P)$ (since we can require P to be the identity on finitely many vectors which form a basis for the image of F). Furthermore, we can require that P almost commutes with the elements of $\mathfrak{F}$. It remains to check that P dominates F. By construction we know that $PF = F$. By computing $\|FP - F\|^2$ we see that $FP = F$ as well. Since $0 \leq F \leq 1$ we get $0 \leq PFP \leq P^2 = P$. Using $PF = FP = F$ we see that $F \leq P$. □

2.8 Definition. Let $\pi : A \to B(H)$ be a $*$-homomorphism. We call π a **quasidiagonal representation** if $\pi(A)$ is a quasidiagonal set of operators.

We say that π is an **essential representation** if $\pi(A) \cap K(H) = \{0\}$. Equivalently we may require that $\pi(A) \cap F(H) = \{0\}$, where $F(H)$ denotes the set of finite rank operators.

2.9 Theorem. *Let A be a unital separable C^*-algebra. Then, the following are equivalent:*

1. *A is QD*
2. *A has a faithful quasidiagonal representation on a separable Hilbert space*
3. *Every faithful unital essential representation of A on a separable Hilbert space is quasidiagonal.*
4. *Lemma 2.3: For every finite subset $\mathfrak{F} \subset A_{\leq 1}$ and $\epsilon > 0$ there exists a representation $\pi : A \to B(H)$ and a finite rank projection $P \in B(H)$ such that for all $a \in \mathfrak{F}$:*

$$\left[\begin{array}{l} \|\pi(a)P - P\pi(a)\| < \epsilon, \\ \|P\pi(a)P\| \geq \|a\| - \epsilon \end{array} \right.$$

Proof. See [Voi91] Theorem 1. □

3 Homotopy invariance of quasidiagonality

3.1 Definition. Let $\sigma_0, \sigma_1 : A \to B$ be $*$-homomorphisms. We say that σ_0 and σ_1 are **homotopic** if there exists a family $\{\sigma_t\}_{t\in(0,1)}$ of $*$-homomorphisms, such that

$$[0,1] \to B : t \mapsto \sigma_t(a)$$

is continuous for all $a \in A$.

This is equivalent to the existence of a $*$-homomorphism $A \to C([0,1]) \otimes B$ with σ_0 as left endpoint and σ_1 as right endpoint.

3.2 Definition. In these notes we will write $A \leq B$ if there exist $*$-homomorphisms $\pi : A \to B$ and $\sigma : B \to A$ such that $\sigma \circ \pi \sim_h \mathrm{id}_A$. If both $A \leq B$ and $B \leq A$ holds, we say that A and B are **homotopy equivalent**.

3.3 Remark. The aim of the rest of the text is to prove the following theorem: If $A \leq B$ and B is QD, then A is also QD. Thus, the notion of quasidiagonality has a strong topological flavor. The following lemma is just another technicality which is needed for the proof.

3.4 Lemma. *Let A be a C^*-algebra, $\epsilon > 0$ and $f \in C([0,1])$ with $f(0) = 0$. Then there exists a $\delta > 0$ such that for all $e, a \in A_{\leq 1}$ with $e \in A_+$ we have*

$$\|ea - ae\| < \delta \quad \Rightarrow \quad \|f(e)a - af(e)\| < \epsilon.$$

Proof. See [Arv77]. □

3.5 Remark. Before we start, we give two more remarks:

(1) If C is separable, then C admits a faithful state. The associated GNS representation $\pi : C \to B(H)$ then represents C faithfully on a separable Hilbert space. To make the representation essential we consider the (still faithful) inflation $\pi^\infty : C \to B(H^\infty) : c \mapsto (\pi(c))_{n\in\mathbb{N}}$ where H^∞ is the countable direct ℓ^2-sum of copies of H.

(2) If H is some Hilbert space and $a \in B(H)$ then, for $\epsilon > 0$ there is a finite rank projection $Q \in B(H)$ with $\|QaQ\| \geq \|a\| - \epsilon$. To see this, let $\psi \in H_{\leq 1}$ with $\|a\psi\| \geq \|a\| - \epsilon$. By considering the projection onto the subspace spanned by ψ and $a\psi$ we see that $\|QaQ\psi\| = \|a\psi\| \geq \|a\| - \epsilon$ and therefore, $\|QaQ\| \geq \|a\| - \epsilon$. Obviously, we can achieve the same result to hold simultaneously for finitely many operators.

3.6 Theorem ([Voi91])**.** *Let $\sigma_0, \sigma_1 : B \to C$ be homotopic $*$-homomorphisms such that σ_0 is injective and $\sigma_1(B)$ is QD. Then B is QD.*

Proof. We may assume that B, C are separable and unital. Furthermore, we can require that the σ_t are unital for $t \in [0,1]$. By (1) we may furthermore assume that C is faithfully and essentially represented on $B(K)$ where K is separable.

In order to prove that B is QD (see Lemma 2.3) , we let $\mathfrak{F} \subset B_{\leq 1}$ be finite and $\epsilon > 0$. We will show that there exists a representation $\pi : B \to B(H)$ and a finite rank projection $P \in B(H)$ such that for all $a \in \mathfrak{F}$:

$$\left[\begin{array}{l} \|\pi(a)P - P\pi(a)\| < \epsilon, \\ \|P\pi(a)P\| \geq \|a\| - \epsilon \end{array} \right. \tag{$\star$}$$

Let $Q \in B(K)$ be a finite rank projection such that

$$\|Q\sigma_0(b)Q\| \geq \|b\| - \epsilon \qquad (b \in \mathfrak{F}).$$

By (2) and the fact that σ_0 is isometric, such a finite rank projection exists. Now, choose $0 < \delta < \frac{\epsilon}{10}$ such that Lemma 3.4 holds with $f(t) = \sqrt{t}$ and such that for all $a, e \in B(K)_{\leq 1}$ with $e \geq 0$:

$$\Big\| [e, a] \Big\| \leq 4\delta \quad \Rightarrow \quad \Big\| [f(e), a] \Big\| \leq \frac{\epsilon}{10} \tag{$*$}$$

Find $n \in \mathbb{N}$ such that

$$\|\psi_{j+1}(b) - \psi_j(b)\| \le \delta$$

for $j = 0, 1, \cdots, n-1,\ b \in \mathfrak{F}$ and $\psi_j = \sigma_{\frac{j}{n}}$.

Indeed, at this point we can already write down what our representation π will be. With

$$H := \bigoplus_{j=0}^{n} K,$$

we define

$$\pi : B \to B(H) : b \mapsto (\psi_j(b))_{j=0}^{n}.$$

However, before we go on we must first do some more work to find the desired projection $P \in B(H)$ which satisfies $(\star)$.

We choose positive finite rank operators $Q \le F_1 \le F_2 \le \cdots \le F_n \le 1$ such that

$$\Big\|[F_j, \psi_j(b)]\Big\| < \delta$$

for $j = 0, 1, 2, \cdots, n$ and $b \in \mathfrak{F}$. The existence of such F_j follows from Corollary 1.7. To apply the corollary one has to look at the separable C*-algebra A generated by the compacts on K together with all elements $\psi_j(b)$ for $j = 0, 1, \cdots, n$ and $b \in \mathfrak{F}$. The corollary gives also that $F_{j+1}F_j = F_j$.

Since $\sigma_1(B)$ is QD and faithfully, essentially represented on K we know from Theorem 2.9 that $\sigma_1(B)$ is a quasidiagonal set of operators. Therefore we can replace F_n by a finite rank projection. See Lemma 2.7.

Now, we can define an operator

$$V : K \to H : x \mapsto F_0^{\frac{1}{2}}(x) \oplus (F_1 - F_0)^{\frac{1}{2}}(x) \oplus \cdots \oplus (F_n - F_{n-1})^{\frac{1}{2}}(x).$$

To simplify notation define $G_0 = F_0$ and $G_j = F_j - F_{j-1}$ for $j = 1, 2, \cdots, n$. Then we may write

$$V(x) = (G_j^{\frac{1}{2}}(x))_{j=0}^{n}.$$

Now, note that $V^*V = F_n$ since

$$\langle V^*Vx, y\rangle = \langle Vx, Vy\rangle = \sum_{j=0}^{n} \langle G_j^{\frac{1}{2}}(x), G_j^{\frac{1}{2}}(y)\rangle = \sum_{j=0}^{n} \langle G_j(x), y\rangle = \langle F_n x, y\rangle.$$

The last equality follows by telescoping argument. Since F_n is a projection we see that V is a partial isometry and hence we get a finite rank projection

$$P := VV^* \in B(H).$$

An important step is now to compute the matrix of P (as an element of $M_{n+1}(B(K))$). To simplify notation we will index such an $(n+1) \times (n+1)$ matrix by entries (i, j) where $i, j = 0, 1, 2, \cdots, n$. A short computation shows that

$$(P)_{i,j} = G_i^{\frac{1}{2}} G_j^{\frac{1}{2}}.$$

Furthermore, if $|i-j| \geq 2$ we see that

$$(F_i - F_{i-1})(F_j - F_{j-1}) = 0$$

by using that $F_j F_i = F_i F_j = F_i$ if $i < j$. Therefore, P has a tridiagonal shape:

$$P = \begin{pmatrix} F_0 & G_0^{\frac{1}{2}} G_1^{\frac{1}{2}} & 0 & 0 & \cdots & 0 \\ G_1^{\frac{1}{2}} G_0^{\frac{1}{2}} & G_1 & G_1^{\frac{1}{2}} G_2^{\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & G_2^{\frac{1}{2}} G_1^{\frac{1}{2}} & G_2 & G_2^{\frac{1}{2}} G_3^{\frac{1}{2}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G_{n-1}^{\frac{1}{2}} G_{n-2}^{\frac{1}{2}} & G_{n-1} & G_{n-1}^{\frac{1}{2}} G_n^{\frac{1}{2}} \\ 0 & 0 & \cdots & 0 & G_n^{\frac{1}{2}} G_{n-1}^{\frac{1}{2}} & G_n \end{pmatrix}$$

We will now check that P satisfies ($\star$): Indeed,

$$\|P\pi(b)P\| \geq \|F_0\psi_0(b)F_0\| \geq \|Q\sigma_0(b)Q\| \geq \|b\| - \epsilon \qquad (b \in \mathfrak{F}).$$

Here, we use the the norm of of an element in $M_{n+1}(B(K))$ is greater than or equal to the norm of all its matrix entries and $(P\pi(b)P)_{0,0} = F_0\psi_0(b)F_0 \geq Q\sigma_0(b)Q$.

Now, we check that P almost commutes with $\pi(b)$ for $b \in \mathfrak{F}$. This requires a few unpleasant calculations: First, note that

$$\left\|[\psi_j(b), G_j]\right\| \leq \left\|[\psi_j(b), F_j]\right\| + \left\|[\psi_j(b), F_{j-1}]\right\| \leq 4\delta.$$

By our assumption in $(*)$ we see that for $j = 0, 1, \cdots, n$:

$$\left\|[G_j^{\frac{1}{2}}, \psi_j(b)]\right\| \leq \frac{\epsilon}{10}. \qquad (**)$$

We now have to look at the norm of the following elements:

$$[P, \pi(b)]_{i,j} = G_i^{\frac{1}{2}} G_j^{\frac{1}{2}} \psi_j(b) - \psi_i(b) G_i^{\frac{1}{2}} G_j^{\frac{1}{2}}.$$

After adding and subtracting the terms $G_i^{\frac{1}{2}}\psi_i(b)G_j^{\frac{1}{2}}$ and $G_i^{\frac{1}{2}}\psi_j(b)G_j^{\frac{1}{2}}$ we see that

$$[P, \pi(b)]_{i,j} = G_i^{\frac{1}{2}}[G_j^{\frac{1}{2}}, \psi_j(b)] + [G_i^{\frac{1}{2}}, \psi_i(b)]G_j^{\frac{1}{2}} + G_i^{\frac{1}{2}}\big(\psi_j(b) - \psi_i(b)\big)G_j^{\frac{1}{2}}.$$

Since all the operators $G_i^{\frac{1}{2}}$ have norm less or equal to one, we see that

$$\begin{aligned} \left\|[P, \pi(b)]_{i,j}\right\| &\leq \left\|[G_j^{\frac{1}{2}}, \psi_j(b)]\right\| + \left\|[G_i^{\frac{1}{2}}, \psi_i(b)]\right\| + \|\psi_j(b) - \psi_i(b)\| \\ &\leq \left(2 \sup_{j=0,1,\cdots,n} \left\|[G_j^{\frac{1}{2}}, \psi_j(b)]\right\|\right) + \|\psi_j(b) - \psi_i(b)\| \\ &\overset{(**)}{\leq} \frac{\epsilon}{5} + \|\psi_j(b) - \psi_i(b)\|. \end{aligned}$$

Using that $[P, \pi(b)]$ has again a tridiagonal shape we may compute

$$\begin{aligned}\left\|[P,\pi(b)]\right\| &\leq 3 \sup_{|i-j|\leq 1} \left\|[P,\pi(b)]_{i,j}\right\| \\ &\leq 3\cdot\frac{\epsilon}{5} + 3 \sup_{|i-j|\leq 1} \|\psi_j(b)-\psi_i(b)\| \\ &\leq \frac{3\epsilon}{5} + 3\delta < \frac{6\epsilon}{10} + \frac{3\epsilon}{10} < \epsilon.\end{aligned}$$

□

3.7 Corollary. *Assume $A \leq B$ and B is QD. Then A is QD.*

Proof. Let $\pi : A \to B$ and $\sigma : B \to A$ be $*$-homomorphisms such that $\sigma \circ \pi \sim_h \mathrm{id}_A$. Define a $*$-homomorphism

$$\eta : A \to A \oplus \pi(A) : a \mapsto \sigma(\pi(a)) \oplus \pi(a).$$

Then $\eta \sim_h \mathrm{id}_A \oplus \pi$ which is injective. Because $\eta(A) \cong \pi(A) \subset B$ and B is QD it follows that $\eta(A)$ is QD. Now, we may apply the previous theorem to $\sigma_0 = \mathrm{id}_A \oplus \pi$ and $\sigma_1 = \eta$. □

3.8 Remark. An immediate consequence is that every contractible C*-algebra (and every subalgebra thereof) is QD. In particular, the cone and suspension of every C*-algebra are QD.

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